

Failure of weak chromatic splitting

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Abstract

Let $n \geq 3$ and let p be any prime. We prove that the canonical map $L_{n-1}S_p \rightarrow L_{n-1}L_{K(n)}S_p$ has no left inverse. The proof follows the source Cartan class in rational degree -3 . A degree-three transfer of constant coefficients factors its localized image through the actual target coefficient map. Witt coefficients remove the extra principal-unit direction. A family of positive modifications detects the resulting target class: quotienting by a trivial line maps this family directly to the coheight-one correspondence. The coefficient comparison uses only low degrees, while rapid convergence supplies the passage to periodic spectra.

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1 Introduction

Let S_p be the p -complete sphere. Write L_m for localization at $K(0) \vee \cdots \vee K(m)$ and $L_{K(m)}$ for localization at Morava K -theory of height m , all at the prime p . The weak chromatic splitting conjecture asks whether the canonical map

$$L_{n-1}S_p \longrightarrow L_{n-1}L_{K(n)}S_p$$

has a left inverse in the stable homotopy category.

Theorem 1.1. *Let $n \geq 3$ and let p be a prime. Then the canonical map*

$$L_{n-1}S_p \longrightarrow L_{n-1}L_{K(n)}S_p$$

has no left inverse.

For odd primes satisfying $p-1 \nmid (n-1)$, Barthel, Mann, Ray, Schlank, Senger, Weinstein, and Zhou construct a split map

$$L_{K(n-1)}S_p \oplus \Sigma^{-1}L_{K(n-1)}S_p \longrightarrow L_{K(n-1)}L_{K(n)}S_p$$

whose unshifted summand is the localized unit [2, Theorem A and Corollary B]. Our obstruction concerns the image of a class from the height- n sphere under localization. Write

$$\begin{aligned} \ell : L_{K(n)}S_p &\longrightarrow L_{K(n-1)}L_{K(n)}S_p, \\ j : L_{K(n-1)}S_p &\longrightarrow L_{K(n-1)}L_{K(n)}S_p \end{aligned} \tag{1.1}$$

for localization and the localized unit, respectively.

Proposition 1.2. *Suppose there are classes*

$$x \in \pi_{-3}L_{K(n)}S_p \otimes \mathbb{Q}, \quad 0 \neq y \in \pi_{-3}L_{K(n-1)}S_p \otimes \mathbb{Q}$$

such that $\ell_(x) = j_*(y)$. Then the canonical map in Theorem 1.1 has no left inverse.*

Proof. Let $i : L_{n-1}S_p \rightarrow L_{n-1}L_{K(n)}S_p$ be the canonical map. Suppose that $r : L_{n-1}L_{K(n)}S_p \rightarrow L_{n-1}S_p$ satisfies $r \circ i \simeq \text{id}$. Let λ denote the L_{n-1} -localization map out of $L_{K(n)}S_p$. Using the natural identifications $L_{K(n-1)}L_{n-1}Z \simeq L_{K(n-1)}Z$, we obtain the following homotopy-commutative diagram; α and β are the two vertical $K(n-1)$ -localization maps:

$$\begin{array}{ccccc} L_{K(n)}S_p & \xrightarrow{\lambda} & L_{n-1}L_{K(n)}S_p & \xrightarrow{r} & L_{n-1}S_p \\ & \searrow \ell & \downarrow \alpha & & \downarrow \beta \\ & & L_{K(n-1)}L_{K(n)}S_p & \xrightarrow{L_{K(n-1)}r} & L_{K(n-1)}S_p \\ & & & \nwarrow j & \end{array}$$

In particular, $L_{K(n-1)}i$ is identified with j . Localizing $r \circ i \simeq \text{id}$ therefore gives

$$(L_{K(n-1)}r) \circ j = \text{id}_{L_{K(n-1)}S_p}.$$

Applying this identity to the assumed equality $\ell_*(x) = j_*(y)$ yields

$$(L_{K(n-1)}r)_*(\ell_*(x)) = (L_{K(n-1)}r)_*(j_*(y)) = y.$$

On the other hand,

$$\pi_{-3}L_{n-1}S_p \otimes \mathbb{Q} = 0,$$

because $L_0L_{n-1}S_p \simeq L_0S_p \simeq H\mathbb{Q}_p$. Thus $r_*(\lambda_*(x)) = 0$. The left triangle gives $\alpha \circ \lambda \simeq \ell$, and the right square gives $(L_{K(n-1)}r) \circ \alpha \simeq \beta \circ r$. Consequently the same element satisfies

$$\begin{aligned} (L_{K(n-1)}r)_*(\ell_*(x)) &= (L_{K(n-1)}r)_*(\alpha_*(\lambda_*(x))) \\ &= \beta_*(r_*(\lambda_*(x))) = 0. \end{aligned}$$

This contradicts $y \neq 0$. □

The rational homotopy of the $K(n)$ -local sphere and the rational stabilizer cohomology of its deformation ring were computed in [3, Theorems A and B]. We use the source Cartan line and its coefficient-unit image from [3, Proposition 3.8.1, Lemma 3.8.5, and Proposition 2.5.1] and identify its image with a nonzero target class. We normalize the source class by the reduced-trace cocycle

$$(X, Y, Z) \longmapsto \text{Trd}(X[Y, Z])$$

on the height- n stabilizer Lie algebra. Three assertions carry the proof. First, the constant-coefficient transfer factors the localized source class through the actual height- $n-1$ coefficient map before completion (Proposition 2.2). Second, after passing through Witt coefficients, the transferred class has no component in the principal-unit direction ((4.1)). Third, its remaining component is nonzero (Proposition 4.1). The natural degree- -3 comparison of Lemma 3.2 then gives the equality required by Proposition 1.2.

The source norm-one geometric error begins in degree n , so it does not affect degree three when $n \geq 4$. At height three, a Witt-coefficient calculation identifies the action of source norm $a \in \mathbb{Z}_p^\times$ on the upper error as a^{-2} . A nontrivial principal-unit weight removes its invariant term. Rapid Morava descent and acyclicity on the specified target coefficient modules supply the comparison with periodic spectra.

The nonzero component is detected on Drinfeld upper half-space. Projecting away the first coordinate quotients the universal height- n positive modification by a trivial line and produces the height- $n-1$ modification. The two classical towers give compact stabilizer torsors for these bundles, with a common arithmetic quotient. Their framed quotient map defines a map from the test family directly into the coheight-one correspondence. Pullback of the transferred class is then the source Cartan class, detected by the last SL_2 block.

The Lubin–Tate and Drinfeld cohomology comparisons used in this detection are due to Barthel, Schlank, Stapleton, and Weinstein [3, Theorem 3.9.4]; Appendices E and F give the low-degree proofs and continuous coefficient compatibilities used here. The filtered meromorphic comparison is due to Gleason, Ivanov, and Zillinger [16, Proposition 5.16]. Appendix D supplies the section-lifting argument and the normalized triangular elimination needed in its proof, and applies it to the coheight-one framed objects of [2, Theorem 2.6.3 and Proposition 2.7.1].

Conventions. Throughout, $n \geq 3$ and p is a prime. For a $\mathbb{Z}_p$ -module M , write $M_{\mathbb{Q}} = M[1/p] = M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$; for homotopy groups this also agrees with tensoring over $\mathbb{Z}$ with $\mathbb{Q}$. For spectra and coefficient complexes, $[1/p]$ denotes ordinary p -inversion in the ambient solid category. The comparison with continuous fixed points is proved where it is used. All group actions and group cohomology are continuous. Derived categories are used as stable infinity-categories, so a natural diagram includes its coherent homotopies. The coefficient objects, their topology, and continuous fixed points are specified in Section 2. The appendices contain the comparison results used in the main proof, together with their

supporting arguments. The section-lifting and triangular-matrix arguments in Appendix D support its filtered comparison; the integral arguments in Appendix F support the two tower comparisons of Appendix E.

2 The actual coefficient transfer

We construct a rational degree-three transfer of constant cohomology and identify its image under the coefficient map of the actual mixed deformation spectrum. The construction uses the uncompleted frame field; completion gives the geometric description used in Section 4.

2.1 The deformation spectra and the frame field

For $i \in \{n, n-1\}$, choose the Honda formal group Γ_i over $\mathbb{F}_p$, with p -series X^{p^i} in a Honda coordinate. We also use Γ_i for the associated connected p -divisible group. Its deformation over $\overline{\mathbb{F}}_p$ includes a specified identification of the special fiber with $(\Gamma_i)_{\overline{\mathbb{F}}_p}$. The Lubin–Tate deformation ring is

$$W(\overline{\mathbb{F}}_p)[[u_1, \dots, u_{i-1}]],$$

and Γ_i^{un} denotes the universal deformation. The coordinates in the two deformation rings are chosen independently. The invariant height ideals are those of [2, Definition 2.1.1 and (2.1.2)–(2.1.3)].

Let

$$D_i = \text{End}_{\overline{\mathbb{F}}_p}((\Gamma_i)_{\overline{\mathbb{F}}_p})[1/p].$$

This is the central division algebra over $\mathbb{Q}_p$ of invariant $1/i$. Its maximal order is $\mathcal{O}_{D_i}$, and $\mathcal{O}_{D_i}^\times$ is the ordinary stabilizer group. The Honda model specifies the arithmetic section of the extended stabilizer

$$G_i = \mathcal{O}_{D_i}^\times \rtimes \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p).$$

These groups act by changing the special-fiber identification and by the arithmetic action on the coefficients. Since the two deformation problems have the same constants, set

$$G_{n,n-1} = G_n \times_{\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)} G_{n-1}.$$

Thus an element of $G_{n,n-1}$ is written (g_n, g_{n-1}, σ) , with a single arithmetic component σ . There is an exact sequence

$$1 \longrightarrow \mathcal{O}_{D_n}^\times \times \mathcal{O}_{D_{n-1}}^\times \longrightarrow G_{n,n-1} \longrightarrow \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \longrightarrow 1. \quad (2.1)$$

We use condensed coefficients to retain continuous actions. A condensed abelian group is a sheaf on profinite sets for finite jointly surjective covers. For a profinite set $U = \varprojlim_j U_j$, with each U_j finite, put

$$\mathbb{Z}[U]^\square = \varprojlim_j \mathbb{Z}[U_j].$$

A condensed abelian group M is solid if $\text{Hom}(\mathbb{Z}[U]^\square, M) \rightarrow M(U)$ is an isomorphism for every extremally disconnected U . We work with solid $\mathbb{Z}_p$ -modules and their derived category; all derived categories are understood as stable infinity-categories. Solidification and the free solid modules are as in [6, §§2.1–2.2, Lemma 2.10].

Fix an uncountable strong limit cardinal large enough for the profinite parameter spaces below. A condensed spectrum may be modeled on the corresponding extremally disconnected profinite sets, with finite coproducts sent to products [12, Definition 11.7]. In an evaluation $M(*)$ or $C(*)$, the symbol $*$ denotes the one-point profinite set. For a condensed spectrum

C , write $\pi_r C$ for its homotopy sheaf and $\pi_r C = \pi_r(C(*))$ for its value at a point. The spectrum is solid if its homotopy sheaves are solid. Evaluation on these test sets preserves limits and Postnikov truncations and detects equivalences.

Let E_i be the solid Morava E -theory of $(\overline{\mathbb{F}}_p, \Gamma_i)$, and put

$$R_i = \pi_0 E_i, \quad R_i(*) = W(\overline{\mathbb{F}}_p)[[u_1, \dots, u_{i-1}]].$$

Its point value is even periodic, with the indicated deformation ring. Its other even homotopy sheaves are the tensor powers of the line of invariant differentials; the odd ones vanish. The solid deformation-spectrum construction and its group action are described in [2, Proposition 4.5.5 and Example 4.6.1]. For the objects used here, Lemma B.1 supplies pointwise descent and its compatibility with profinite parameters. We use $R_i(U)$ and $E_i(U)$ for evaluations, and R_i and E_i for the solid objects.

The height- $n-1$ ideal of $R_n(*)$ is $(p, u_1, \dots, u_{n-2})$. Its height- $n-1$ localization has residue field

$$(R_n(*)/(p, u_1, \dots, u_{n-2}))[u_{n-1}^{-1}] = \overline{\mathbb{F}}_p((u_{n-1})),$$

equipped with its complete valuation topology. The base change of Γ_n^{un} to this field has total height n , connected height $n-1$, and an étale quotient of height one [2, Proposition 2.2.1]. Its connected-part frame functor classifies isomorphisms

$$(\Gamma_{n-1})_A \xrightarrow{\sim} (\Gamma_n^{\text{un}})_A^\circ$$

over $\overline{\mathbb{F}}_p((u_{n-1}))$ -algebras A . The change-of-frame group is $\mathcal{O}_{D_{n-1}}^\times$. By Lemma C.1, this pro-étale torsor is $\text{Spec } L$, where $L/\overline{\mathbb{F}}_p((u_{n-1}))$ is the totally ramified separable Galois Gross–Torii extension with Galois group $\mathcal{O}_{D_{n-1}}^\times$. The lemma recalls Gross’s coefficient argument and establishes its finite-constant-field version; see also [2, Definition 1.2.5 and Example 4.6.3]. It carries the universal frame

$$\alpha : (\Gamma_{n-1})_L \xrightarrow{\sim} (\Gamma_{n,L}^{\text{un}})^\circ.$$

The source stabilizer transports the deformation and its frame. The target element g_{n-1} changes the frame to $\alpha \circ g_{n-1}^{-1}$, and the common arithmetic component acts semilinearly. These operations give the $G_{n,n-1}$ -action on L . The valuation on $\overline{\mathbb{F}}_p((u_{n-1}))$ extends uniquely to L ; its completion $\widehat{L}$ is a perfect characteristic- p field and the action extends continuously by Lemma C.1. Their condensed coefficient rings are

$$L(U) = \text{colim}_j C(U, L_j), \quad \widehat{L}(U) = C(U, \widehat{L}), \quad (2.2)$$

where L_j runs through the finite extensions of $\overline{\mathbb{F}}_p((u_{n-1}))$ in L , with their complete valuation topologies. Thus a section of $L(U)$ lies in one finite stage, whereas $\widehat{L}(U)$ consists of all continuous maps to the completion. The residue field is viewed as the condensed ring $U \mapsto C(U, \overline{\mathbb{F}}_p((u_{n-1})))$. The constant field $\overline{\mathbb{F}}_p$ is discrete, with sections $C(U, \overline{\mathbb{F}}_p)$. All these coefficient rings carry the profinite-parameter structures of [2, Proposition 4.5.1 and Example 4.6.3].

For an ordinary spectrum V , write V^δ for its constant solid spectrum. To retain the continuous arithmetic action on the tensor base, use the finite constant fields and set

$$\mathbb{S}_{\text{sm}} = \text{colim}_{d \geq 1} \mathbb{S}_W(\mathbb{F}_{p^d})^\delta. \quad (2.3)$$

Here the index category is the positive integers ordered by divisibility, and the inclusions are those in $\overline{\mathbb{F}}_p$. The colimit is taken in solid commutative ring spectra. Each stage has

arithmetic action through the finite group $\text{Gal}(\mathbb{F}_{p^d}/\mathbb{F}_p)$. Their compatible actions give a continuous $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ -action on $\mathbb{S}_{\text{sm}}$.

The inclusions of finite constant fields give compatible unital maps

$$\mathbb{S}_W(\mathbb{F}_{p^d})^\delta \longrightarrow E_i, \quad i \in \{n, n-1\},$$

by [1, Theorem 1.1]. They are continuous equivariant maps, with action on the source through the finite arithmetic quotient of G_i . Indeed, on every profinite parameter U , the target $E_i(U)$ is $K(i)$ -local and hence ordinarily p -complete. Antieau's adjunction specifies the map by the constant-field embedding into $(\pi_0 E_i(U)/p)^{\text{perf}}$. On $G_i^a \times U$ these embeddings are the same semilinear constant-field maps; the adjunction retains their composition, faces, and degeneracies. Taking the colimit gives the equivariant cospan

$$L_{K(n-1)}E_n \longleftarrow \mathbb{S}_{\text{sm}} \longrightarrow E_{n-1},$$

with the common arithmetic component of $G_{n,n-1}$.

Define the mixed deformation spectrum by

$$B = L_{K(n-1)}(E_n \otimes_{\mathbb{S}_{\text{sm}}} E_{n-1}).$$

Tensor products and localizations here are taken in solid spectra. This cospan supplies the continuous $G_{n,n-1}$ -action on B . Since $K(n-1)$ -acyclic objects form a tensor ideal, localization of the first factor gives the equivalent presentation

$$B \simeq L_{K(n-1)}((L_{K(n-1)}E_n) \otimes_{\mathbb{S}_{\text{sm}}} E_{n-1}).$$

The two tensor factors give the actual equivariant unital maps

$$L_{K(n-1)}E_n \longrightarrow B \longleftarrow E_{n-1}. \quad (2.4)$$

Their source and target actions factor through G_n and G_{n-1} , respectively.

We record the comparison with the full spherical Witt ring after forgetting the arithmetic action. Spherical Witt vectors preserve colimits in the category of ordinary p -complete commutative ring spectra, since they are the left adjoint of [1, Theorem 1.1]. Consequently

$$\left(\text{colim}_{d \geq 1} \mathbb{S}_W(\mathbb{F}_{p^d}) \right)_p^\wedge \simeq \mathbb{S}_W(\overline{\mathbb{F}}_p).$$

The completion here precedes constant condensation. Its unit is a Moore- p equivalence. Constant condensation is exact and preserves filtered colimits, so

$$\mathbb{S}_{\text{sm}} \longrightarrow \mathbb{S}_W(\overline{\mathbb{F}}_p)^\delta$$

also has cofiber on which multiplication by p is invertible. In the two-sided bar construction for relative tensor products, enlarging the base by this map is a Moore- p equivalence in each simplicial degree, and hence after geometric realization. Positive-height $K(n-1)$ -localization kills its p -invertible cofiber. We therefore obtain a canonical equivalence

$$B \xrightarrow{\sim} L_{K(n-1)}((L_{K(n-1)}E_n) \otimes_{\mathbb{S}_W(\overline{\mathbb{F}}_p)^\delta} E_{n-1}). \quad (2.5)$$

This comparison commutes with both tensor-factor maps.

Proposition 4.5.12 of [2], using the construction and proof of Theorem 4.4.8 there, identifies the right side with the solid deformation spectrum of the framed mixed pair $(L, \Gamma_{n,L}^{\text{un}})$. The two structure maps have a useful moduli description under this identification.

The source map is deformation base change along the frame field extension. The target map classifies the connected part of the universal oriented deformation, tagged by the specified frame $\alpha : (\Gamma_{n-1})_L \simeq (\Gamma_{n,L}^{\text{un}})^\circ$. This is the connected-part tagging used in the coproduct proof of Theorem 4.4.8.

The compatible finite-field taggings above identify the continuous actions as well. On $G_{n,n-1}^a \times U$, the source transports the deformation, the target changes the frame by composition with g_{n-1}^{-1} , and the common arithmetic component acts semilinearly. The two tags and the common orientation in that coproduct proof transform by these same maps. Proposition 4.5.5 and Remark 4.5.6 of [2] retain profinite parameters and their cotensors, so these comparisons respect all faces and degeneracies of the action nerve. Thus the identification with the framed deformation spectrum is $G_{n,n-1}$ -equivariant and preserves the two maps in (2.4). The reduction of $\pi_0 B$ is the specified special fiber L , and the target unit induces a map of complete pairs with residue map $\overline{\mathbb{F}}_p \rightarrow L$:

$$\begin{array}{ccc} R_{n-1} & \longrightarrow & \pi_0 B \\ \downarrow & & \downarrow \\ \overline{\mathbb{F}}_p & \longrightarrow & L. \end{array} \quad (2.6)$$

The kernels of the vertical maps are the invariant height- $n-1$ ideals. On every profinite test set, they are ideals of definition and the corresponding rings are complete and separated. For $\pi_0 B$, the Noetherian witness is

$$(W(\overline{\mathbb{F}}_p)[[u_1, \dots, u_{n-1}]] [u_{n-1}^{-1}])_{(p, u_1, \dots, u_{n-2})}^\wedge. \quad (2.7)$$

The identification of its ideal with the deformation kernel is [2, Definition 4.4.1 and Proposition 4.4.4]. Flat relatively perfect deformation base change preserves the ideal and its residue [2, Proposition 4.2.4 and Lemma 4.2.12]; completeness is supplied by Remark 4.2.7 and Proposition 4.4.2 there. These facts, together with the profinite-parameter descent in Appendix B, apply to $(\pi_0 B)(U)$ for every U .

2.2 The quotient group and source descent

For the unit group of $\mathbb{Z}_p$, use the decomposition

$$\mathbb{Z}_p^\times = \Delta_p \times \Gamma, \quad \begin{cases} \Delta_p = \mu_{p-1}, & \Gamma = 1 + p\mathbb{Z}_p, & p \neq 2, \\ \Delta_2 = \{\pm 1\}, & \Gamma = 1 + 4\mathbb{Z}_2, & p = 2. \end{cases}$$

Thus Δ_p is the finite unit factor and $\Gamma \simeq \mathbb{Z}_p$ is the principal-unit factor used throughout the paper. Let Nrd_n be reduced norm on $\mathcal{O}_{D_n}^\times$ and let pr_Γ project to Γ . The quotient homomorphism

$$G_{n,n-1} \longrightarrow \Gamma \times G_{n-1}, \quad (g_n, g_{n-1}, \sigma) \longmapsto (\text{pr}_\Gamma \text{Nrd}_n(g_n), (g_{n-1}, \sigma))$$

is well defined because reduced norm is arithmetic invariant. Its kernel is

$$N = \{(g_n, 1, 1) : \text{Nrd}_n(g_n) \in \Delta_p\}.$$

It gives the exact sequence

$$1 \longrightarrow N \longrightarrow G_{n,n-1} \longrightarrow \Gamma \times G_{n-1} \longrightarrow 1. \quad (2.8)$$

The subgroup N acts trivially on $\overline{\mathbb{F}}_p$ and R_{n-1} . It contains the ordinary source norm-one group, with quotient Δ_p . For the Honda models, the determinant comparison over $\mathbb{F}_{p^2}$ has arithmetic discrepancy of order at most two [2, §2.7]. This lies in Δ_p and disappears under pr_Γ , giving the quotient coordinates used in [2, §5.2].

For a profinite group G and a solid coefficient complex M , write $R\Gamma_{\mathrm{cts}}(G, M)$ for derived continuous fixed points. Set

$$H_{\mathrm{cts}}^a(G, M) = \mathcal{H}^a(R\Gamma_{\mathrm{cts}}(G, M))(*).$$

Write $D(\mathrm{Solid}_{\mathbb{Z}_p})$ for the derived category of solid $\mathbb{Z}_p$ -modules. For a locally profinite group P , let $D(\mathrm{Solid}_P)$ be the derived category of solid $\mathbb{Z}_p$ -modules with continuous P -action, equipped with the cohomological t -structure on the underlying complexes. For a solid spectrum C , C^{hG} denotes continuous homotopy fixed points, computed by the continuous action-cochain totalization. Cotensors are given by profinite parameters: $C^U(V) = C(U \times V)$, with the analogous convention for coefficient complexes. Iterated fixed points for (2.8) identify $G_{n,n-1}$ -fixed points with $\Gamma \times G_{n-1}$ -fixed points after taking N -fixed points.

The source-descent equivalences are those of [2, Proposition 4.6.13 and Corollary 4.6.15]. We record their compatibility with the two actual unital maps.

Lemma 2.1. *The source map in (2.4) induces unital equivalences*

$$L_{K(n-1)}E_n \xrightarrow{\sim} B^{h\mathcal{O}_{D_{n-1}}^\times}, \quad L_{K(n-1)}L_{K(n)}S_p \xrightarrow{\sim} (B^{hG_{n,n-1}})(*).$$

Under the second equivalence, the map $E_n \rightarrow L_{K(n-1)}E_n \rightarrow B$ induces the localization map $\ell : L_{K(n)}S_p \rightarrow L_{K(n-1)}L_{K(n)}S_p$. The map induced by $E_{n-1} \rightarrow B$ on sphere fixed points is the actual target unit $j : L_{K(n-1)}S_p \rightarrow L_{K(n-1)}L_{K(n)}S_p$.

Proof. The first augmentation is the source descent of [2, Proposition 4.6.13]. Lemma A.6 below records its residue descent for the full ordinary target group, the adic lift, and compatibility with the residual G_n -action. The bounded-above Postnikov argument in Lemma B.1 passes from these homotopy-module augmentations to the spectral augmentation. Taking the remaining G_n -fixed points gives the second equivalence by [2, Corollary 4.6.15]. The solid Morava descent equivalences

$$(E_i^{hG_i})(*) \simeq L_{K(i)}S_p, \quad i \in \{n, n-1\},$$

are the unital equivalences of [2, Proposition 4.6.9]. All these augmentations are induced by the actual source and target maps in (2.4). They therefore identify the source map on sphere fixed points with ℓ and the target map with j . $\square$

2.3 The transfer and its coefficient image

For a solid characteristic- p ring F , define

$$\begin{aligned} \mathrm{ASW}_m(F) &= \mathrm{fib}(\varphi - 1 : W_m(F) \longrightarrow W_m(F)), \\ \mathrm{ASW}(F) &= \varprojlim_m \mathrm{ASW}_m(F). \end{aligned}$$

Here W_m is length- m Witt vectors and φ is the equal-length endomorphism induced by Frobenius on F . The fiber is placed in cohomological degrees zero and one. The Verschiebung filtration has successive additive quotients F , so these complexes are solid for $F = \overline{\mathbb{F}}_p, L, \hat{L}$. Their constant units come from the Artin–Schreier–Witt sequence.

Proposition 2.2. *There is a commutative square of solid complexes with continuous $\Gamma \times G_{n-1}$ -action*

$$\begin{array}{ccc} \mathbb{Z}_p & \longrightarrow & R\Gamma_{\text{cts}}(N, \text{ASW}(L)) \\ \downarrow & & \downarrow \\ R_{n-1} & \longrightarrow & R\Gamma_{\text{cts}}(N, \pi_0 B). \end{array} \quad (2.9)$$

The top and left arrows are constant units. The bottom arrow is adjoint to the coefficient map of $E_{n-1} \rightarrow B$, and the right arrow is induced by the unit-compatible map of Proposition 2.6.

After taking $\Gamma \times G_{n-1}$ -fixed points, the top arrow induces the isomorphism

$$u_L^3 : H_{\text{cts}}^3(\Gamma \times G_{n-1}, \mathbb{Z}_p)_{\mathbb{Q}} \xrightarrow{\sim} H_{\text{cts}}^3(G_{n,n-1}, \text{ASW}(L))_{\mathbb{Q}}$$

of Lemma 2.7. Define the transfer in degree three by

$$\begin{aligned} \tau : H_{\text{cts}}^3(G_{n,n-1}, \mathbb{Z}_p)_{\mathbb{Q}} &\longrightarrow H_{\text{cts}}^3(G_{n,n-1}, \text{ASW}(L))_{\mathbb{Q}} \\ &\xrightarrow{(u_L^3)^{-1}} H_{\text{cts}}^3(\Gamma \times G_{n-1}, \mathbb{Z}_p)_{\mathbb{Q}}, \end{aligned} \quad (2.10)$$

where the first map is the coefficient unit. For every $z \in H_{\text{cts}}^3(G_{n,n-1}, \mathbb{Z}_p)_{\mathbb{Q}}$, its coefficient image in $H_{\text{cts}}^3(G_{n,n-1}, \pi_0 B)_{\mathbb{Q}}$ is the image of $\tau(z)$ under

$$\begin{aligned} H_{\text{cts}}^3(\Gamma \times G_{n-1}, \mathbb{Z}_p)_{\mathbb{Q}} &\longrightarrow H_{\text{cts}}^3(\Gamma \times G_{n-1}, R_{n-1})_{\mathbb{Q}} \\ &\longrightarrow H_{\text{cts}}^3(G_{n,n-1}, \pi_0 B)_{\mathbb{Q}}. \end{aligned} \quad (2.11)$$

The last arrow is specified by the actual target unit $E_{n-1} \rightarrow B$. Replacing L by $\hat{L}$ in (2.10) gives the same transfer.

We prove the proposition by constructing the unit-compatible map $\text{ASW}(L) \rightarrow \pi_0 B$ and computing the constant unit in degree three after N - and $\Gamma \times G_{n-1}$ -fixed points. The completion comparison identifies this calculation with cohomology of the punctured Banach–Colmez space.

Lemma 2.3. *The canonical map $\text{ASW}(L) \rightarrow \text{ASW}(\hat{L})$ is an equivalence of solid complexes with continuous $G_{n,n-1}$ -action, compatible with the constant unit.*

Proof. At length one, it suffices to show that $\varphi - 1$ is an automorphism of $\hat{L}/L$ as an additive solid quotient. Evaluation on extremally disconnected profinite sets is exact and detects equivalences. Fix such a set U . For $x \in C(U, \hat{L})$, compactness and density give a locally constant function y with finitely many values in L such that $e = x - y$ has a uniform strictly positive valuation. The values of y lie in a single finite complete stage L_j . The uniformly convergent series

$$z = - \sum_{r \geq 0} e^{p^r}$$

satisfies $(\varphi - 1)z = e$. It follows that $\varphi - 1$ is surjective on $\hat{L}(U)/L(U)$.

For injectivity, suppose $(\varphi - 1)x \in L(U)$. Choose an approximation y as above and a finite complete stage L_a containing its values and the section $(\varphi - 1)x$. Then $b = (\varphi - 1)(x - y)$ belongs to $C(U, L_a)$ and has uniform positive valuation. The same convergent series applied to b gives $z \in C(U, L_a)$ with $(\varphi - 1)z = b$. The function $x - y - z$ is continuous and $\mathbb{F}_p$ -valued, hence belongs to $L(U)$. Thus x belongs to $L(U)$.

For $m > 1$, the additive sequence

$$0 \longrightarrow W_{m-1}(F) \xrightarrow{V} W_m(F) \longrightarrow F \longrightarrow 0$$

commutes with equal-length Frobenius. Taking fibers of $\varphi - 1$ and inducting proves the comparison at every finite length; derived inverse limits give the result. All comparison maps are induced by $L \rightarrow \hat{L}$. They retain the $G_{n,n-1}$ -action and the constant unit. $\square$

2.4 A henselian construction of the coefficient unit

For a ring A , let $\text{FinEt}(A)$ be the site of finite étale A -schemes, with jointly surjective covers. An object is denoted by its coordinate algebra S , so a coefficient functor is covariant in algebra maps. Write $R\Gamma_{\text{fét}}(A, -)$ for derived sections on this site.

Lemma 2.4. *For every ring A and A -module M , the sheaf $\widetilde{M}(S) = M \otimes_A S$ satisfies*

$$M \xrightarrow{\sim} R\Gamma_{\text{fét}}(A, \widetilde{M}).$$

The augmentation is natural in ring maps and module maps.

Proof. The sheaf property is faithfully flat descent for modules. Every object of the site is affine and quasi-compact, and finite étale maps have open images. Thus finite covering families are cofinal among all covering families. For a finite cover $\{\text{Spec } T_i \rightarrow \text{Spec } S\}$, put $T = \prod_i T_i$. Then $S \rightarrow T$ is finite faithfully flat. Since tensoring with $M \otimes_A S$ preserves finite products, the Čech complex of this cover is the Amitsur complex

$$M \otimes_A S \longrightarrow M \otimes_A T \longrightarrow M \otimes_A (T \otimes_S T) \longrightarrow \cdots.$$

It is exact in positive degrees and has degree-zero term $M \otimes_A S$ by [7, Tag 03OA]. Finite étale objects are closed under all the fiber products in these complexes. Apply the Čech-to-derived criterion [7, Tag 03F9, Lemma 21.10.9], taking all site objects and all finite covering families. It proves the required vanishing on every object, and hence the displayed augmentation equivalence. $\square$

Lemma 2.5. *Let (A, J) be a complete separated adic pair with $p \in J$, and put $F = A/J$. There is a natural additive derived map*

$$\alpha_A : \text{ASW}(F) \longrightarrow A$$

whose composite with the constant unit $\mathbb{Z}_p \rightarrow \text{ASW}(F)$ is the derived p -complete unit of A . It is natural for maps of complete pairs carrying the specified ideal into the specified ideal.

Proof. Completeness makes (A, J) henselian [7, Tag 0ALJ]. Reduction induces an equivalence

$$\text{FinEt}(A) \simeq \text{FinEt}(F)$$

by [7, Tag 09ZL]. This is an equivalence of sites: the ideal lies in the Jacobson radical, and surjectivity of a finite étale covering is detected after reduction. For a finite étale F -algebra S , let A_S be its lift with the specified residual identification. The groupoid of such lifts is contractible, since reduction is fully faithful as well as essentially surjective.

On $\text{FinEt}(A)$, consider $\mathcal{O}_A \otimes_{\mathbb{Z}}^L \mathbb{Z}/p^m$. It is represented by the two-term structure-sheaf complex $[\mathcal{O}_A \xrightarrow{p^m} \mathcal{O}_A]$. Transport it to $\text{FinEt}(F)$ and call the resulting complex $\mathcal{A}_m$; thus

$$\mathcal{A}_m(S) = A_S \otimes_{\mathbb{Z}}^L \mathbb{Z}/p^m.$$

Lemma 2.4, applied on $\text{FinEt}(A)$ to each term, gives the natural augmentation equivalence

$$A \otimes_{\mathbb{Z}}^L \mathbb{Z}/p^m \xrightarrow{\sim} R\Gamma_{\text{fét}}(F, \mathcal{A}_m). \quad (2.12)$$

On $\text{FinEt}(F)$, the Artin–Schreier–Witt sequence is exact. Its equations are solved by successive monic Artin–Schreier equations with derivative -1 , and hence by finite étale

covers. Lemma 2.4 gives acyclicity of $\mathcal{O}_F$; its finite Verschiebung filtration then gives acyclicity of $W_m(\mathcal{O}_F)$. Consequently

$$R\Gamma_{\text{fét}}(F, \mathbb{Z}/p^m) \simeq \text{ASW}_m(F).$$

Take derived sections of the sheaf unit

$$(\mathbb{Z}/p^m)_{\text{const}} \simeq \mathbb{Z}_{\text{const}} \otimes^L \mathbb{Z}/p^m \longrightarrow \mathcal{A}_m$$

and invert the augmentation (2.12). This defines

$$\text{ASW}_m(F) \longrightarrow A \otimes_{\mathbb{Z}}^L \mathbb{Z}/p^m. \quad (2.13)$$

The composite with the constant coaugmentation is the unit $\mathbb{Z}/p^m \rightarrow A \otimes^L \mathbb{Z}/p^m$.

The quotient tower A/J^r is surjective, so ordinary completeness and separation identify

$$A \simeq \varprojlim_r A/J^r.$$

Since p^r kills A/J^r , each term is derived p -complete. Derived p -complete complexes are closed under limits, whence

$$A \simeq \varprojlim_m (A \otimes_{\mathbb{Z}}^L \mathbb{Z}/p^m). \quad (2.14)$$

This is also the one-generator derived-completion criterion of [7, §15.93, Lemmas 15.93.14 and 15.93.17]. Taking the derived inverse limit of (2.13) defines α_A .

The lifting equivalence, the structure-sheaf unit, and the augmentation maps commute with base change of complete pairs. The residual identifications specify the lifts and all their comparison isomorphisms. The construction also commutes with length reduction, so these diagrams retain their coherences after inverse limits. Their constant coaugmentations show that α_A preserves the specified derived unit. $\square$

Proposition 2.6. *There is an additive map of solid complexes with continuous $G_{n,n-1}$ -action*

$$\alpha : \text{ASW}(L) \longrightarrow \pi_0 B$$

compatible with the constant units $\mathbb{Z}_p \rightarrow \text{ASW}(L)$ and $\mathbb{Z}_p \rightarrow \pi_0 B$.

Proof. For every profinite set U , apply Lemma 2.5 to the complete pair $(\pi_0 B)(U)$ with its height- $n-1$ ideal and residue $L(U)$. The map $\alpha_{(\pi_0 B)(U)}$ is natural in U . Derived condensed sheafification gives a map between the corresponding condensed complexes; its endpoints are the solid complexes $\text{ASW}(L)$ and $\pi_0 B$, so it is a map in the full derived solid subcategory. In particular, evaluation on extremally disconnected parameter sets identifies (2.14) with the derived completion of the solid coefficient ring.

Apply the same construction to the coaction maps on $G_{n,n-1}^a \times U$ for every $a \geq 0$. Naturality for these maps respects multiplication, unit, all faces, and all degeneracies of the action groupoid. Thus the map has the asserted continuous $G_{n,n-1}$ -equivariance. The constant coaugmentations in (2.13) give unit compatibility in this same equivariant derived category. $\square$

2.5 The norm quotient and descent of constants

Let

$$S_n = \ker(\mathrm{Nrd}_n : \mathcal{O}_{D_n}^\times \longrightarrow \mathbb{Z}_p^\times).$$

Thus $S_n \triangleleft N$ and $N/S_n = \Delta_p$.

The geometric inputs are the norm-one quotient and its characteristic- p cohomology [2, Proposition 2.7.1(5), Proposition 3.6.9, and Theorem 3.6.10]. We use them to prove the rational degree-three comparison at every prime. At height three the additional Witt weight is established in Appendix G.

Lemma 2.7. *The constant unit induces an isomorphism*

$$\begin{aligned} u_L^3 : H_{\mathrm{cts}}^3(\Gamma \times G_{n-1}, \mathbb{Z}_p)_\mathbb{Q} &\xrightarrow{\sim} H_{\mathrm{cts}}^3(G_{n,n-1}, \mathrm{ASW}(L))_\mathbb{Q} \\ &\xrightarrow{\sim} H_{\mathrm{cts}}^3(G_{n,n-1}, \mathrm{ASW}(\widehat{L}))_\mathbb{Q}. \end{aligned} \quad (2.15)$$

The first map is obtained from $\mathbb{Z}_p \rightarrow R\Gamma_{\mathrm{cts}}(N, \mathrm{ASW}(L))$ by taking $\Gamma \times G_{n-1}$ -fixed points. The second is the completion comparison. Write u_L^3 for the composite.

Proof. Since $\overline{\mathbb{F}}_p$ is algebraically closed, $\varphi - 1$ on $W_m(\overline{\mathbb{F}}_p)$ is surjective with kernel $\mathbb{Z}/p^m$. This also holds with profinite parameters: a section of $W_m(C(U, \overline{\mathbb{F}}_p))$ has finitely many values, whose preimages can be chosen on clopen pieces of U . The inverse limit of these kernels is the solid module of continuous $\mathbb{Z}_p$ -valued functions. Consequently the specified unit identifies $\mathrm{ASW}(\overline{\mathbb{F}}_p)$ with $\mathbb{Z}_p$. By Lemma 2.3, it remains to compute the unit for $\widehat{L}$.

The geometric quotient is

$$[\mathrm{Spd} \widehat{L}/S_n] \simeq \mathrm{BC}(-1/(n-1))^\times$$

by Corollary D.9. The Banach–Colmez space classifies extensions of $\mathcal{O}(1/(n-1))$ by $\mathcal{O}$ on the Fargues–Fontaine curve. Its punctured locus consists of extensions that are nonzero at every geometric point.

We record the residual source action on this space. Fix the determinant isomorphism

$$\delta : \det \mathcal{O}(1/n) \xrightarrow{\sim} \det \mathcal{O}(1/(n-1))$$

constructed in Lemma D.8. For a normalized extension

$$0 \longrightarrow \mathcal{O} \xrightarrow{i} \mathcal{O}(1/n) \xrightarrow{q} \mathcal{O}(1/(n-1)) \longrightarrow 0$$

with class e , an ordinary element $(g_n, g_{n-1}, 1)$ replaces q by $g_{n-1}qg_n^{-1}$. Determinant compatibility gives

$$\begin{aligned} i' &= \frac{\mathrm{Nrd}_{n-1}(g_{n-1})}{\mathrm{Nrd}_n(g_n)} g_n i, \\ e' &= \frac{\mathrm{Nrd}_n(g_n)}{\mathrm{Nrd}_{n-1}(g_{n-1})} (g_{n-1}^{-1})^* e. \end{aligned}$$

Thus a source element of norm a , with identity target component, acts by the pure dilation $e \mapsto ae$. The target central scalar aI_{n-1} acts by $e \mapsto a^{-n}e$, and the diagonal pair (aI_n, aI_{n-1}) acts trivially.

Lemma D.11, applied through Corollary D.9, gives the constant-unit triangle

$$\overline{\mathbb{F}}_p \longrightarrow R\Gamma_{\mathrm{cts}}(S_n, \widehat{L}) \longrightarrow T_{\mathrm{top}}[-n], \quad (2.16)$$

where $T_{\mathrm{top}} = \mathcal{H}^n R\Gamma_{\mathrm{cts}}(S_n, \widehat{L})$ is a one-dimensional solid $\overline{\mathbb{F}}_p$ -module. The triangle retains the actual residual $G_{n,n-1}/S_n$ -action; its arithmetic part is semilinear.

Put

$$\mathcal{C}_m = \text{cofib}(W_m(\overline{\mathbb{F}}_p) \longrightarrow R\Gamma_{\text{cts}}(S_n, W_m(\widehat{L}))).$$

The additive Verschiebung sequence gives equivariant triangles

$$\mathcal{C}_{m-1} \longrightarrow \mathcal{C}_m \longrightarrow \mathcal{C}_1 \quad (m \geq 2), \quad \mathcal{C}_1 = T_{\text{top}}[-n].$$

Hence $\mathcal{C}_m \in D^{\geq n}$ for every m . Taking Frobenius fibers in the diagram of constant units gives

$$\text{cofib}(\mathbb{Z}/p^m \longrightarrow R\Gamma_{\text{cts}}(S_n, \text{ASW}_m(\widehat{L}))) \simeq \text{fib}(\varphi - 1 : \mathcal{C}_m \rightarrow \mathcal{C}_m).$$

Fibers and derived inverse limits preserve $D^{\geq n}$. Thus the Artin–Schreier–Witt unit error satisfies

$$\mathcal{C}_S = \text{cofib}(\mathbb{Z}_p \longrightarrow R\Gamma_{\text{cts}}(S_n, \text{ASW}(\widehat{L}))) \in D^{\geq n}. \quad (2.17)$$

All these constructions preserve the actual residual action.

For the further quotient by $\Delta_p = N/S_n$, let

$$\mathcal{C}_N = \text{cofib}(\mathbb{Z}_p \longrightarrow R\Gamma_{\text{cts}}(N, \text{ASW}(\widehat{L}))).$$

Iterated fixed points and composition of the two units give a natural triangle with $\Gamma \times G_{n-1}$ -action

$$\text{cofib}(\mathbb{Z}_p \longrightarrow R\Gamma_{\text{cts}}(\Delta_p, \mathbb{Z}_p)) \longrightarrow \mathcal{C}_N \longrightarrow R\Gamma_{\text{cts}}(\Delta_p, \mathcal{C}_S). \quad (2.18)$$

For odd p , averaging over $\Delta_p = \mu_{p-1}$ is exact, so the first term is zero and $\mathcal{C}_N \in D^{\geq n}$. For $p = 2$, the first term has lower bound two, degree-two cohomology $\mathbb{F}_2$, and degree-three cohomology zero: indeed

$$\mathcal{H}^{2r} R\Gamma_{\text{cts}}(C_2, \mathbb{Z}_2) = \mathbb{Z}/2 \quad (r \geq 1), \quad \mathcal{H}^{2r+1} R\Gamma_{\text{cts}}(C_2, \mathbb{Z}_2) = 0.$$

Since $\mathcal{C}_S \in D^{\geq 3}$, we obtain

$$\mathcal{C}_N \in D^{\geq 2}, \quad \mathcal{H}^2 \mathcal{C}_N = \begin{cases} 0, & p \text{ odd}, \\ \mathbb{F}_2, & p = 2. \end{cases}$$

If $n \geq 4$, (2.18) gives $\mathcal{H}^3 \mathcal{C}_N = 0$. If $n = 3$, Lemma G.1 identifies $\mathcal{C}_S$ with a $\mathbb{Z}_p$ -line in degree three, whose actual residual source norm action is

$$a \in \mathbb{Z}_p^\times \longmapsto a^{-2} \text{ on } \mathcal{H}^3 \mathcal{C}_S. \quad (2.19)$$

The same triangle gives an equivariant injection

$$\mathcal{H}^3 \mathcal{C}_N \hookrightarrow (\mathcal{H}^3 \mathcal{C}_S)^{\Delta_p}.$$

Take $a = 1 + p$ for odd p , and $a = 5$ for $p = 2$. This is a source norm element of Γ and acts on the line by $a^{-2} \neq 1$. The line is torsion-free, so its invariant submodule under Γ is zero. It follows that $\mathcal{H}^3 \mathcal{C}_N$ has no $\Gamma \times G_{n-1}$ -invariants for every $n \geq 3$.

Only the first two Postnikov terms can affect the desired degree-three unit. They form the triangle

$$(\mathcal{H}^2 \mathcal{C}_N)[-2] \longrightarrow \tau_{\leq 3} \mathcal{C}_N \longrightarrow (\mathcal{H}^3 \mathcal{C}_N)[-3].$$

Derived fixed points preserve cohomological lower bounds, so $\tau_{\geq 4} \mathcal{C}_N$ does not contribute in degrees two or three. The long exact sequence, together with $(\mathcal{H}^3 \mathcal{C}_N)^{\Gamma \times G_{n-1}} = 0$, shows that the degree-two and degree-three cohomology of $R\Gamma_{\text{cts}}(\Gamma \times G_{n-1}, \mathcal{C}_N)$ are subquotients of

$$H_{\text{cts}}^0(\Gamma \times G_{n-1}, \mathcal{H}^2 \mathcal{C}_N), \quad H_{\text{cts}}^1(\Gamma \times G_{n-1}, \mathcal{H}^2 \mathcal{C}_N),$$

respectively. These groups vanish for odd p and are killed by two for $p = 2$. Finally, the long exact sequence of the actual constant unit gives (2.15) after inverting p . $\square$

Proof of Proposition 2.2. The coefficient map $R_{n-1} \rightarrow \pi_0 B$ is $G_{n,n-1}$ -equivariant, and N acts trivially on R_{n-1} . Its adjoint is the bottom map in (2.9). The right map is induced by Proposition 2.6. Under the inflation–fixed-point adjunction, both paths around the square correspond to the same equivariant derived unit $\mathbb{Z}_p \rightarrow \pi_0 B$. This proves the square with its continuous action.

Take $\Gamma \times G_{n-1}$ -fixed points and use iterated fixed points for (2.8). In degree three, Lemma 2.7 makes the top map invertible after inverting p . Composing the square with its inverse gives (2.11). The completion equivalence of Lemma 2.3 commutes with the constant unit, and hence identifies u_L^3 with $u_{\widehat{L}}^3$. It therefore leaves the transfer (2.10) unchanged. $\square$

2.6 The geometric identity for the transfer

We work on perfectoid spaces of characteristic p with the v -topology. For a perfectoid field F , $\mathrm{Spd} F$ is the diamond of $\mathrm{Spa}(F, F^+)$, with F^+ its valuation ring. Locally profinite groups are regarded as group v -sheaves. For an action of G on a v -sheaf Z , the quotient $[Z/G]$ classifies G -torsors with equivariant maps to Z .

The terminal v -sheaf on this characteristic- p site is $\mathrm{Spd} \mathbb{F}_p$: it assigns a singleton to every test perfectoid space. With the trivial G -action on this terminal object, define

$$BG = [\mathrm{Spd} \mathbb{F}_p / G].$$

Thus, on a test perfectoid space S , the groupoid $BG(S)$ consists of G -torsors on S for the v -topology.

Write $R\Gamma_v$ for solid-enhanced derived v -sections. Its evaluation at a profinite U computes sections after product with U . Constant $\mathbb{Z}_p$ coefficients mean the derived inverse limit of the finite constant sheaves $\mathbb{Z}/p^m$. We set $H_v^a(Z, M) = \mathcal{H}^a(R\Gamma_v(Z, M))(*)$.

Lemma 2.8. *For every profinite U and $m \geq 1$, there is a natural equivalence*

$$\begin{aligned} R\Gamma_v(\mathrm{Spd} \widehat{L} \times U, \mathbb{Z}/p^m) \\ \simeq \mathrm{fib}(\varphi - 1 : W_m(C(U, \widehat{L})) \longrightarrow W_m(C(U, \widehat{L}))). \end{aligned}$$

It preserves the constant units, Frobenius, length reduction, and the continuous $G_{n,n-1}$ -action. For every closed subgroup $V \subset G_{n,n-1}$, it induces

$$R\Gamma_v([\mathrm{Spd} \widehat{L}/V], \mathbb{Z}_p) \simeq R\Gamma_{\mathrm{cts}}(V, \mathrm{ASW}(\widehat{L})),$$

compatibly with restriction of V .

Proof. The product with U is the affinoid perfectoid of $(C(U, \widehat{L}), C(U, \widehat{L}^+))$. Its $\mathcal{O}^b$ -cohomology is $C(U, \widehat{L})$ in degree zero, by [2, Theorem 3.2.1(ii), (iii) and its proof]; Remark 3.2.3 there supplies the solid enhancement. The Verschiebung sequences imply

$$R\Gamma_v(\mathrm{Spd} \widehat{L} \times U, W_m(\mathcal{O}^b)) \simeq W_m(C(U, \widehat{L})).$$

The Artin–Schreier–Witt fiber sequence gives the asserted finite-length comparison. All maps come from the coefficient sheaves, so their naturality preserves the unit and the Witt operations.

Apply these comparisons to $V^a \times U$ for all $a \geq 0$. They identify the faces and degeneracies of the torsor action groupoid with those of continuous cochains. Totalization gives the comparison for the quotient stack. Finally take derived inverse limits in m ; both derived sections and totalization preserve these limits. $\square$

Define

$$\mathcal{Y} = [\mathrm{Spd} \widehat{L}/G_{n,n-1}], \quad BG_{n,n-1} \xleftarrow{f_{n,n-1}} \mathcal{Y} \xrightarrow{f_{\mathrm{target}}} B(\Gamma \times G_{n-1}). \quad (2.20)$$

The first map classifies the tautological $G_{n,n-1}$ -torsor and the second its extension of structure group along the quotient homomorphism in (2.8). If $\mathrm{pr}_n : G_{n,n-1} \rightarrow G_n$ is source projection, the actual source map is $f_n = B \mathrm{pr}_n \circ f_{n,n-1} : \mathcal{Y} \rightarrow BG_n$. These descriptions specify the source and target torsors whose restrictions will be compared in Section 4.

For a profinite group G , let

$$\kappa_G : R\Gamma_{\mathrm{cts}}(G, \mathbb{Z}_p) \longrightarrow R\Gamma_v(BG, \mathbb{Z}_p)$$

be the canonical comparison obtained by applying the constant unit on the base of the classifying stack to its action groupoid. Equivalently, the unit $\mathbb{Z}_p \rightarrow R\Gamma_v(\mathrm{Spd} \mathbb{F}_p, \mathbb{Z}_p)$ induces the map on continuous action-cochain totalizations, with profinite parameters retained. This specifies the map to absolute v -cohomology used below.

Proposition 2.9. *The constant-unit map induces an isomorphism*

$$f_{\mathrm{target}}^* \kappa_{\Gamma \times G_{n-1}} : H_{\mathrm{cts}}^3(\Gamma \times G_{n-1}, \mathbb{Z}_p)_{\mathbb{Q}} \xrightarrow{\sim} H_v^3(\mathcal{Y}, \mathbb{Z}_p)_{\mathbb{Q}}.$$

For every $z \in H_{\mathrm{cts}}^3(G_{n,n-1}, \mathbb{Z}_p)_{\mathbb{Q}}$, the transfer of Proposition 2.2 satisfies

$$f_{\mathrm{target}}^* \kappa_{\Gamma \times G_{n-1}}(\tau(z)) = f_{n,n-1}^* \kappa_{G_{n,n-1}}(z) \quad \text{in } H_v^3(\mathcal{Y}, \mathbb{Z}_p)_{\mathbb{Q}}. \quad (2.21)$$

For a source class $z_n \in H_{\mathrm{cts}}^3(G_n, \mathbb{Z}_p)_{\mathbb{Q}}$, the right side for $z = \mathrm{pr}_n^* z_n$ is $f_n^* \kappa_{G_n}(z_n)$.

Proof. Lemma 2.8 identifies $R\Gamma_v(\mathcal{Y}, \mathbb{Z}_p)$ with $R\Gamma_{\mathrm{cts}}(G_{n,n-1}, \mathrm{ASW}(\widehat{L}))$. Under this equivalence, $f_{n,n-1}^* \kappa_{G_{n,n-1}}$ is induced by the constant unit $\mathbb{Z}_p \rightarrow \mathrm{ASW}(\widehat{L})$. The map $f_{\mathrm{target}}^* \kappa_{\Gamma \times G_{n-1}}$ is the constant-unit map

$$R\Gamma_{\mathrm{cts}}(\Gamma \times G_{n-1}, \mathbb{Z}_p) \longrightarrow R\Gamma_{\mathrm{cts}}(\Gamma \times G_{n-1}, R\Gamma_{\mathrm{cts}}(N, \mathrm{ASW}(\widehat{L}))).$$

Its degree-three rationalization is u_L^3 of Lemma 2.7. Composing the source unit with the inverse of this isomorphism gives the completed transfer. Lemma 2.3 identifies it with (2.10), proving (2.21). The source-class assertion follows from naturality under pr_n . $\square$

3 Degree-three classes and homotopy

The coefficient factorization will be used in rational homotopy degree -3 . We first compare this degree with continuous cohomology, compatibly with the two maps into B . We then define the source Cartan class and record the Witt cohomology calculation used in the assembly. All spectra and actions are the solid refinements fixed above.

3.1 Continuous fixed points in degree -3

All inversions of p in this section take place in the ambient category of solid spectra or solid modules. For a solid module M with continuous G -action, $R\Gamma_{\mathrm{cts}}(G, M)$ denotes internal derived invariants; its cohomology evaluated at the one-point test set is $H_{\mathrm{cts}}^s(G, M)$.

The comparisons below use two descent properties. The ordinary target stabilizer is acyclic on the specified homotopy modules of B , whereas the source Morava descent complex is quickly converging. Together they give the canonical equivalences

$$(X^{hG})[1/p] \xrightarrow{\sim} (X[1/p])^{hG} \quad \begin{cases} (X, G) = (E_i, G_i), & i \in \{n-1, n\}, \\ (X, G) = (B, G_{n,n-1}), \end{cases} \quad (3.1)$$

proved in Lemma A.1. After this comparison, a central procyclic group isolates the degree-zero homotopy module.

Lemma 3.1 (Central scalar action). *Let G be a profinite group, M a solid $\mathbb{Z}_p$ -module with continuous G -action, and $c \in G$ central. If c acts on M by $\lambda \in \mathbb{Z}_p^\times$, then*

$$(\lambda - 1) \operatorname{id}_{R\Gamma_{\text{cts}}(G, M)} \simeq 0.$$

If $\lambda \neq 1$, this complex vanishes after inversion of p . If $\lambda \neq 1$ and multiplication by p is already invertible on M , the complex itself vanishes.

Proof. On homogeneous continuous bar cochains, coefficient action by c is represented by

$$(T_c f)(g_0, \dots, g_q) = f(cg_0, \dots, cg_q).$$

Equivariance identifies this operator with multiplication by λ . Centrality makes the usual prism operator equivariant:

$$(hf)(g_0, \dots, g_{q-1}) = \sum_{j=0}^{q-1} (-1)^j f(g_0, \dots, g_j, cg_j, \dots, cg_{q-1}).$$

The bar differential gives $dh + hd = T_c - \operatorname{id}$. Each insertion is continuous and commutes with the diagonal G -action, so this homotopy also holds internally, on every profinite parameter. The bar resolution computes derived invariants by [6, Lemmas 2.11(3) and 2.14]. Finally, a nonzero $\lambda - 1$ is a power of p times a unit; after inversion of p it is both invertible and nullhomotopic. $\square$

Lemma 3.2. *There are natural rational degree-three comparisons*

$$\begin{aligned} \pi_{-3}(E_i^{hG_i})[1/p] &\simeq H_{\text{cts}}^3(G_i, R_i)[1/p] & (i \in \{n-1, n\}), \\ \pi_{-3}(B^{hG_{n,n-1}})[1/p] &\simeq H_{\text{cts}}^3(G_{n,n-1}, \pi_0 B)[1/p]. \end{aligned}$$

These are induced by internal comparisons before evaluation at the one-point test set. They commute with the actual maps $E_n \rightarrow B$ and $E_{n-1} \rightarrow B$ and inflation along the two projections from $G_{n,n-1}$.

Proof. Put

$$a = \begin{cases} 1 + p, & p \text{ odd}, \\ 5, & p = 2. \end{cases}$$

For E_i , let $C_i \subset G_i$ be the closed central group generated by aI_i . For B , let $C_B \subset G_{n,n-1}$ be the closed central group generated by $(aI_n, aI_{n-1}, 1)$. Each group is isomorphic to $\mathbb{Z}_p$; the two projections identify C_B with C_n and C_{n-1} , respectively.

Multiplication by a is a natural automorphism of a p -divisible group and its deformations. In the mixed framed deformation problem it preserves the connected part and commutes with the connected-part frame [2, Example 4.6.3]. The diagonal scalar therefore acts trivially on $\pi_0 B$, and aI_i acts trivially on R_i . On invariant differentials, multiplication by a acts by a . The even homotopy modules are tensor powers of that line; odd homotopy modules vanish [2, §4.1 and Examples 4.6.1–4.6.3]. Thus the scalar on degree $2r$ is a^r or a^{-r} , according to the periodicity convention. It differs from 1 whenever $r \neq 0$.

Fix one of the pairs (X, G) in the statement, and denote its central group by C . Lemma 3.1 shows that every nonzero-degree Eilenberg–Mac Lane layer of $X[1/p]$ has zero C -fixed points.

The group $C \simeq \mathbb{Z}_p$ has internal cohomological dimension one (Lemma A.4). Lemma A.5, applied in each output degree to $\tau_{\geq 1}(X[1/p])$ and $\tau_{\leq -1}(X[1/p])$, therefore gives

$$(\tau_{\geq 1}(X[1/p]))^{hC} \simeq 0, \quad (\tau_{\leq -1}(X[1/p]))^{hC} \simeq 0.$$

The natural truncation maps give a zigzag of equivalences

$$(X[1/p])^{hC} \longrightarrow (\tau_{\leq 0}(X[1/p]))^{hC} \longleftarrow (H(\pi_0 X[1/p]))^{hC}. \quad (3.2)$$

It is equivariant for the residual G/C -action, obtained by iterated continuous fixed points. Taking its fixed points and using (3.1) identifies the rational degree -3 of X^{hG} with

$$H_{\text{cts}}^3(G, \pi_0 X[1/p]) \simeq H_{\text{cts}}^3(G, \pi_0 X)[1/p].$$

The last comparison is Lemma A.3.

All arrows in (3.1) are the canonical inversion maps, and all arrows in (3.2) are truncation maps. The actual maps into B respect the central groups under the two projections. They therefore give maps of these zigzags, including their residual quotient actions. This proves the asserted naturality. Under Lemma 2.1, the source map is ℓ of (1.1); the other map is the actual target map used in the assembly. $\square$

3.2 Witt coefficients and the source Cartan class

The arithmetic quotient of G_i acts on $W(\overline{\mathbb{F}}_p)$ and $W(\overline{\mathbb{F}}_p)[1/p]$ by its usual semilinear action.

The stabilizer cohomology calculation in the next lemma is [3, Proposition 3.8.1 and Lemma 3.8.5]. We also record its consequence for the principal-unit factor.

Lemma 3.3. *For $i \in \{n-1, n\}$, there is an isomorphism of graded $\mathbb{Q}_p$ -algebras*

$$\begin{aligned} H_{\text{cts}}^*(G_i, W(\overline{\mathbb{F}}_p))[1/p] &\simeq H_{\text{cts}}^*(G_i, W(\overline{\mathbb{F}}_p)[1/p]) \\ &\simeq \Lambda_{\mathbb{Q}_p}(z_1, z_3, \dots, z_{2i-1}), \quad |z_j| = j. \end{aligned}$$

In particular, $H_{\text{cts}}^2(G_{n-1}, W(\overline{\mathbb{F}}_p))[1/p] = 0$. Let $\text{pr} : \Gamma \times G_{n-1} \rightarrow G_{n-1}$ be the projection and $\iota : G_{n-1} \rightarrow \Gamma \times G_{n-1}$ the inclusion $g \mapsto (1, g)$. With trivial Γ -action on $W(\overline{\mathbb{F}}_p)$, the map

$$\text{pr}^* : H_{\text{cts}}^3(G_{n-1}, W(\overline{\mathbb{F}}_p))[1/p] \xrightarrow{\sim} H_{\text{cts}}^3(\Gamma \times G_{n-1}, W(\overline{\mathbb{F}}_p))[1/p]$$

has inverse ι^ .*

Proof. We first justify the passage from $W(\overline{\mathbb{F}}_p)$ to $W(\overline{\mathbb{F}}_p)[1/p]$. For a profinite group P , the trivial solid $\mathbb{Z}_p[[P]]$ -module has a resolution by compact projectives [6, Theorem 3.2]. Applying the external Hom functor to this resolution makes every cochain term commute with filtered colimits. Filtered colimits of solid modules are exact. Consequently external continuous cohomology in each fixed degree commutes with the filtered colimit $M[1/p] = \varinjlim (M \xrightarrow{p} M \xrightarrow{p} \dots)$. In particular,

$$H_{\text{cts}}^s(G_i, W(\overline{\mathbb{F}}_p))[1/p] \simeq H_{\text{cts}}^s(G_i, W(\overline{\mathbb{F}}_p)[1/p]).$$

The second isomorphism in the statement is [3, Lemma 3.8.5(2)]. Its arithmetic input is

$$W(\overline{\mathbb{F}}_p)^{\widehat{\mathbb{Z}}} = \mathbb{Z}_p, \quad H_{\text{cts}}^r(\widehat{\mathbb{Z}}, W(\overline{\mathbb{F}}_p)) = 0 \quad (r > 0)$$

[3, Lemma 3.8.5(1)]. Together with the ordinary stabilizer computation of [3, Proposition 3.8.1], it gives precisely the displayed exterior algebra. That algebra has one degree-one generator and all other generators have odd degree at least three; its degree-two part is zero.

Finally, choose a topological generator γ of $\Gamma \simeq \mathbb{Z}_p$. The length-one completed augmentation resolution computes continuous Γ -invariants by multiplication by $\gamma - 1$. Since Γ acts trivially on $W(\overline{\mathbb{F}}_p)$, taking its invariants before or after those of G_{n-1} gives

$$\begin{aligned} R\Gamma_{\text{cts}}(\Gamma \times G_{n-1}, W(\overline{\mathbb{F}}_p)) \\ \simeq R\Gamma_{\text{cts}}(G_{n-1}, W(\overline{\mathbb{F}}_p)) \oplus R\Gamma_{\text{cts}}(G_{n-1}, W(\overline{\mathbb{F}}_p))[-1]. \end{aligned}$$

Here the cohomological shift $[-1]$ contributes $H_{\text{cts}}^{s-1}(G_{n-1}, W(\overline{\mathbb{F}}_p))$ in degree s . Projection to the first summand is restriction along ι , and its inclusion is inflation along pr . In degree three, the second summand vanishes after inversion of p by the degree-two calculation. This proves the final assertion. $\square$

Let D_n be the division algebra of invariant $1/n$. Its Lie algebra is the $\mathbb{Q}_p$ -vector space D_n with bracket $[X, Y] = XY - YX$. Let $\text{Trd} : D_n \rightarrow \mathbb{Q}_p$ be the reduced trace. The reduced-trace Cartan cochain is

$$\kappa_n(X, Y, Z) = \text{Trd}(X[Y, Z]).$$

Cyclicity of reduced trace makes it alternating. The identity $\text{Trd}([X, Y]Z) = \text{Trd}(X[Y, Z])$ and the Jacobi identity give $d\kappa_n = 0$ in the Chevalley–Eilenberg complex with trivial coefficients.

The Cartan line and its nonzero coefficient-unit image are the rational stabilizer and sphere inputs of [3, Proposition 3.8.1, Lemma 3.8.5, and Proposition 2.5.1]. We fix the reduced-trace normalization and use the natural comparison of Lemma 3.2.

Lemma 3.4. *The cocycle κ_n determines a nonzero class*

$$\vartheta_n \in H_{\text{cts}}^3(G_n, \mathbb{Z}_p)[1/p]. \quad (3.3)$$

Its coefficient-unit image defines a nonzero homotopy class

$$x_n \in \pi_{-3}L_{K(n)}S_p[1/p]$$

under Lemma 3.2 and $(E_n^{hG_n})() \simeq L_{K(n)}S_p$. Both the class and its coefficient-unit image are induced by integral cohomology classes after multiplication by a power of p .*

Proof. Choose a sufficiently small open normal subgroup $U \subset \mathcal{O}_{D_n}^\times$. The Lazard comparison identifies

$$H_{\text{cts}}^3(U, \mathbb{Q}_p) \simeq H_{\text{Lie}}^3(D_n, \mathbb{Q}_p),$$

naturally for restriction to smaller such subgroups and for conjugation. This is the comparison used in [3, Proposition 3.8.1 and its proof]. The Cartan cocycle is invariant under conjugation. Finite-group descent with $\mathbb{Q}_p$ -coefficients therefore extends its class uniquely from U to $\mathcal{O}_{D_n}^\times$. Compatibility with smaller open subgroups makes this extension independent of U .

The Lie class is nonzero. After extending scalars to a splitting field of D_n , it is the trace cocycle on $\mathfrak{gl}_n$. Restrict to a 2×2 block with basis $H_0 = \text{diag}(1, -1)$, $E = E_{12}$, and $F = E_{21}$. It has value

$$\text{Tr}(H_0[E, F]) = 2.$$

For every alternating two-cochain ω on this $\mathfrak{sl}_2$,

$$(d\omega)(H_0, E, F) = -2\omega(E, F) - 2\omega(F, E) - \omega(H_0, H_0) = 0.$$

Thus the restriction is not a coboundary. Scalar extension and restriction preserve coboundaries, so the original Lie class, and hence the group class, is nonzero.

The ordinary stabilizer computation is

$$H_{\text{cts}}^*(\mathcal{O}_{D_n}^\times, \mathbb{Q}_p) \simeq \Lambda_{\mathbb{Q}_p}(z_1, z_3, \dots, z_{2n-1})$$

[3, Proposition 3.8.1]. In particular, its degree-two part vanishes and its degree-three part is the Cartan line. The arithmetic action preserves this line: Frobenius acts on the division algebra by conjugation by a uniformizer, and reduced trace is invariant under conjugation. The Lyndon–Hochschild–Serre spectral sequence for

$$1 \longrightarrow \mathcal{O}_{D_n}^\times \longrightarrow G_n \longrightarrow \widehat{\mathbb{Z}} \longrightarrow 1$$

has at most two columns. Its degree-three restriction map is an isomorphism onto the invariant Cartan line, since the possible degree-two term on the kernel is zero. Thus the chosen Cartan class on $\mathcal{O}_{D_n}^\times$ has a unique extension to $H_{\text{cts}}^3(G_n, \mathbb{Q}_p)$, and this extension is nonzero. The filtered-colimit argument in Lemma 3.3, applied to $\mathbb{Z}_p$, identifies this group with $H_{\text{cts}}^3(G_n, \mathbb{Z}_p)[1/p]$ and defines ϑ_n in (3.3).

The constant inclusion into $W(\overline{\mathbb{F}}_p)[1/p]$ sends this class to its scalar extension on the ordinary stabilizer. The arithmetic calculation of [3, Lemma 3.8.5 and its proof] identifies the degree-three edge for Witt coefficients with the same Cartan line over $\mathbb{Q}_p$. The constant Cartan class therefore has nonzero image in $H_{\text{cts}}^3(G_n, W(\overline{\mathbb{F}}_p))[1/p]$. The continuous G_n -equivariant additive splitting $W(\overline{\mathbb{F}}_p) \rightarrow R_n$ of [3, Proposition 2.5.1] shows that its further image under the coefficient unit is nonzero in $H_{\text{cts}}^3(G_n, R_n)[1/p]$. Lemma 3.2 gives the asserted nonzero class x_n . Finally, every element of a group localized at p is an integral element divided by a power of p . The coefficient unit carries such an integral representative to an integral representative of its image. $\square$

4 A test family for the transfer

Inflate the source Cartan class to $G_{n,n-1}$ and apply the transfer. Its target coefficient image is

$$\eta = \text{unit}_{R_{n-1},*} \tau(\inf_{G_n}^{G_{n,n-1}} \vartheta_n) \in H_{\text{cts}}^3(\Gamma \times G_{n-1}, R_{n-1})[1/p].$$

The action of Γ on R_{n-1} is trivial. The coefficient unit factors through $W(\overline{\mathbb{F}}_p)$, on which Γ also acts trivially. By Lemma 3.3,

$$H_{\text{cts}}^2(G_{n-1}, W(\overline{\mathbb{F}}_p))[1/p] = 0.$$

The two-term continuous resolution for $\Gamma \simeq \mathbb{Z}_p$ therefore gives an isomorphism

$$\text{pr}^* : H_{\text{cts}}^3(G_{n-1}, W(\overline{\mathbb{F}}_p))[1/p] \xrightarrow{\sim} H_{\text{cts}}^3(\Gamma \times G_{n-1}, W(\overline{\mathbb{F}}_p))[1/p].$$

Here pr is projection onto G_{n-1} . Its inverse is restriction along $\iota : G_{n-1} \rightarrow \Gamma \times G_{n-1}$, $g \mapsto (1, g)$. Apply this to the Witt coefficient image of $\tau(\inf_{G_n}^{G_{n,n-1}} \vartheta_n)$ and then map to R_{n-1} . Thus, with

$$b = \iota^* \eta \in H_{\text{cts}}^3(G_{n-1}, R_{n-1})[1/p], \quad \eta = \text{pr}^* b, \quad (4.1)$$

the target coefficient class comes from G_{n-1} .

Proposition 4.1. *The class b in (4.1) is nonzero.*

We will construct a family of adjacent modifications with actual source and target stabilizer torsors. Its source Cartan class survives in completed structure-sheaf cohomology. The target coefficient unit sends b to this class.

4.1 The quotient modification

For $i \in \{n-1, n\}$, write

$$\mathbb{H}^{i-1} = \mathbb{P}_{\mathbb{Q}_p}^{i-1} \setminus \bigcup_{P \text{ a } \mathbb{Q}_p\text{-rational hyperplane}} P$$

for Drinfeld's symmetric space. At a geometric point its homogeneous coordinates have no nonzero $\mathbb{Q}_p$ -linear relation. We use column coordinates, with $g \in \mathrm{GL}_i(\mathbb{Q}_p)$ acting by $x \mapsto gx$.

An S -point of $\mathbb{H}^{i-1, \diamond}$ includes an untilt $S^\sharp$ over $\mathbb{Q}_p$. Let $\iota_S : S^\sharp \hookrightarrow X_S$ be its divisor on the relative Fargues–Fontaine curve. The modification description of [4, Theorem 3.6.1, equations (3.6.2)–(3.6.3)] gives a universal sequence

$$0 \longrightarrow \mathcal{O}_{X_S}^i \longrightarrow \mathcal{V}_i \longrightarrow (\iota_S)_* \mathcal{L}_i \longrightarrow 0, \quad (4.2)$$

where $\mathcal{L}_i$ is a line bundle on $S^\sharp$ and $\mathcal{V}_i$ is isoclinic of slope $1/i$ at every geometric point. The trivial term has its standard frame. The bundle $\mathcal{O}_{X_S}(1/i)$ associated with the height- i Honda isocrystal has rank i and degree one.

The dual negative modification is determined by a quotient $\mathcal{O}_{S^\sharp}^i \twoheadrightarrow \mathcal{L}'_i$ and uses the contragredient action. The positive column convention above is obtained from this convention by $g \mapsto (g^{-1})^T$. We use the corresponding version of the moduli identifications throughout this section.

Coordinate projection defines a morphism

$$q : \mathbb{H}^{n-1} \longrightarrow \mathbb{H}^{n-2}, \quad [x_1 : \cdots : x_n] \longmapsto [x_2 : \cdots : x_n].$$

Its center is excluded from $\mathbb{H}^{n-1}$. A rational linear relation among the projected coordinates would be a rational linear relation among the original coordinates, so the image lies in $\mathbb{H}^{n-2}$.

Let $\mathrm{SL}_2(\mathbb{Z}_p)$ act on the source and target through the last 2×2 blocks, respectively:

$$M \longmapsto \mathrm{diag}(I_{n-2}, M), \quad M \longmapsto \mathrm{diag}(I_{n-3}, M).$$

When $n = 3$, I_{n-3} is the empty block. The projection q is equivariant for these actions, and the source action fixes e_1 . Inverse transpose preserves both block actions, so the same equivariance holds in the dual convention. Define

$$\mathcal{C} = [\mathbb{H}^{n-1, \diamond} / \mathrm{SL}_2(\mathbb{Z}_p)].$$

Lemma 4.2. *On $\mathcal{C}$, quotienting the universal middle bundle by its first trivial line gives an exact sequence of vector bundles*

$$0 \longrightarrow \mathcal{O}e_1 \longrightarrow \mathcal{V}_n \xrightarrow{q^\vee} \mathcal{V}_{n-1} \longrightarrow 0. \quad (4.3)$$

Here $\mathcal{V}_{n-1}$ is the equivariant pullback by q of the universal height- $n-1$ middle bundle. Its modification has the same untilt divisor and cokernel line as (4.2) in rank n .

Proof. Work first on $\mathbb{H}^{n-1, \diamond}$. Dualizing the positive modification gives

$$0 \longrightarrow \mathcal{V}_n^\vee \longrightarrow \mathcal{O}^n \longrightarrow (\iota_S)_* \mathcal{L}' \longrightarrow 0.$$

Restrict the last map to the last $n-1$ coordinates. Since these coordinates have no common zero, the restriction is surjective. It is precisely the quotient defining the negative modification at $q(x)$. Consequently the two negative modifications form a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{V}_{n-1}^\vee & \longrightarrow & \mathcal{O}^{n-1} & \longrightarrow & (\iota_S)_* \mathcal{L}' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{V}_n^\vee & \longrightarrow & \mathcal{O}^n & \longrightarrow & (\iota_S)_* \mathcal{L}' \longrightarrow 0. \end{array}$$

The middle inclusion omits the first coordinate. The snake lemma gives

$$0 \longrightarrow \mathcal{V}_{n-1}^\vee \longrightarrow \mathcal{V}_n^\vee \longrightarrow \mathcal{O} \longrightarrow 0.$$

All three terms are vector bundles: the first is the universal negative modification pulled back along q , and the quotient is $\mathcal{O}$. Duality gives (4.3), with kernel inclusion the original $\mathcal{O}_{e_1}$.

The cokernel line in the positive modification is recovered from $\mathcal{L}'$ by duality and the normal line of ι_S ; both are unchanged [4, proof of Theorem 3.6.1]. This identifies the entire rank- $n-1$ positive modification. The construction commutes with base change and with the $\mathrm{SL}_2(\mathbb{Z}_p)$ -action, which fixes the kernel frame e_1 . It therefore descends to $\mathcal{C}$. $\square$

4.2 The actual stabilizer torsors

Write $\mathrm{LT}_i = \mathrm{Spf} R_i(*)$ and $\mathrm{LT}_{i,\eta}$ for its rigid generic fiber. The moduli equivalence of [4, Theorem 3.6.1] is

$$[\mathbb{H}^{i-1,\diamond} / \mathrm{GL}_i(\mathbb{Z}_p)] \simeq [\mathrm{LT}_{i,\eta}^\diamond / G_i]. \quad (4.4)$$

The source modification and its quotient define two moduli maps:

$$\begin{aligned} m_n : \mathcal{C} &\longrightarrow [\mathbb{H}^{n-1,\diamond} / \mathrm{GL}_n(\mathbb{Z}_p)] \simeq [\mathrm{LT}_{n,\eta}^\diamond / G_n], \\ m_{n-1} : \mathcal{C} &\longrightarrow [\mathbb{H}^{n-2,\diamond} / \mathrm{GL}_{n-1}(\mathbb{Z}_p)] \simeq [\mathrm{LT}_{n-1,\eta}^\diamond / G_{n-1}]. \end{aligned}$$

The first arrow is induced by the last-block inclusion $\mathrm{SL}_2(\mathbb{Z}_p) \subset \mathrm{GL}_n(\mathbb{Z}_p)$. The second is induced by q and the last-block action in rank $n-1$. For $i \in \{n, n-1\}$, the quotient projection $\mathrm{LT}_{i,\eta}^\diamond \rightarrow [\mathrm{LT}_{i,\eta}^\diamond / G_i]$ is the tautological G_i -torsor. Define its pullback by the Cartesian square

$$\begin{array}{ccc} P_i & \xrightarrow{\tilde{m}_i} & \mathrm{LT}_{i,\eta}^\diamond \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{m_i} & [\mathrm{LT}_{i,\eta}^\diamond / G_i]. \end{array}$$

The fiber product is taken in v -stacks; explicitly,

$$P_i = \mathcal{C} \times_{[\mathrm{LT}_{i,\eta}^\diamond / G_i]} \mathrm{LT}_{i,\eta}^\diamond.$$

A section of P_i over a chart $S \rightarrow \mathcal{C}$ is a lift $S \rightarrow \mathrm{LT}_{i,\eta}^\diamond$ together with an identification of its image in the quotient stack with $m_i|_S$. The lift supplies the constant-field choice and Honda marking of a Lubin–Tate representative of the family. The classical tower supplies pro-étale covers on which these lifts exist, and G_i acts simply transitively on their choices. We regard P_i as a left torsor; $\tilde{m}_i$ is then equivariant for the given left action on $\mathrm{LT}_{i,\eta}^\diamond$. The classifying map p_i forgets the moduli point and retains this torsor:

$$p_i : \mathcal{C} \xrightarrow{m_i} [\mathrm{LT}_{i,\eta}^\diamond / G_i] \longrightarrow BG_i.$$

To pull a constant group-cohomology class a back to $\mathcal{C}$, we use $p_i^* \kappa_{G_i}(a)$, where κ_{G_i} is the canonical comparison to constant v -cohomology defined before Proposition 2.9.

To identify their common arithmetic quotient, introduce the quasi-isogeny groups

$$G_i^0 = D_i^\times \rtimes \mathrm{Gal}(\overline{\mathbb{F}}_p / \mathbb{F}_p), \quad i \in \{n-1, n\}.$$

For a family $\mathcal{V}_i$ as above, its arithmetic isocrystal-frame torsor records a choice of constants $S \rightarrow \mathrm{Spd} \overline{\mathbb{F}}_p$ and an isomorphism $\mathcal{O}_{X_S}(1/i) \xrightarrow{\sim} \mathcal{V}_i$. With the constants fixed, its frame group

is $D_i^\times$. The relative bundle classification [4, Proposition 3.5.1 and the paragraph following it] identifies this arithmetic frame construction with the stack $[\mathrm{Spd} \overline{\mathbb{F}}_p / G_i^0]$.

The moduli equivalence (4.4) identifies this torsor with

$$P_i^0 = G_i^0 \times^{G_i} P_i.$$

Here $(a, p) \sim (ag^{-1}, gp)$ and G_i^0 acts on the first factor. Indeed, [4, Proposition 3.3.2] identifies the height-zero Rapoport–Zink component with LT_i . The natural Rapoport–Zink–shtuka isomorphism [4, Theorem 3.5.3] identifies its quasi-isogeny frame with the middle-bundle frame and commutes with the map to $\mathrm{Spd} W(\overline{\mathbb{F}}_p)[1/p]$. Passing to the quotient gives exactly the moduli identification used in (4.4).

Let

$$\mathcal{T}_{\mathrm{arith}} = \mathcal{C} \times_{\mathrm{Spd} \overline{\mathbb{F}}_p} \mathrm{Spd} \overline{\mathbb{F}}_p$$

be the torsor of choices of the constant field. The quotient $P_i^0 / D_i^\times$ records only these choices. Since extension of the frame group does not change its arithmetic quotient, there are specified identifications

$$P_i / \mathcal{O}_{D_i}^\times \simeq P_i^0 / D_i^\times \simeq \mathcal{T}_{\mathrm{arith}}. \quad (4.5)$$

These maps forget the frame and retain the constants. On a chart, if the map of constants is $\iota : \overline{\mathbb{F}}_p \rightarrow R$, then the left arithmetic action of σ replaces it by $\iota \circ \sigma^{-1}$ in both ranks [4, §3.3]. Thus these identifications use the same specified arithmetic torsor, including its action map.

Form the fiber product

$$P = P_n \times_{\mathcal{T}_{\mathrm{arith}}} P_{n-1}.$$

By (4.5), this is a torsor under $G_{n,n-1} = G_n \times_{\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)} G_{n-1}$.

4.3 Arithmetic equivariance of the surjection description

The Honda models over $\mathbb{F}_p$ specify the arithmetic splittings of the extended stabilizers G_i and the extended quasi-isogeny groups G_i^0 . Their common extended quasi-isogeny group is

$$G_{n,n-1}^0 = G_n^0 \times_{\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)} G_{n-1}^0.$$

On a perfectoid space S supplied with a map to $\mathrm{Spd} \overline{\mathbb{F}}_p$, write $\mathcal{E}_i = \mathcal{O}_{X_S}(1/i)$, where X_S is the relative Fargues–Fontaine curve. Let $\mathcal{Q}$ be the v -sheaf over $\mathrm{Spd} \overline{\mathbb{F}}_p$ of surjections $\mathcal{E}_n \twoheadrightarrow \mathcal{E}_{n-1}$. The Honda models give a semilinear arithmetic action on these bundles. Thus, with transport along this descent datum understood, $G_{n,n-1}^0$ acts by

$$(g_n, g_{n-1}, \sigma) \cdot q = g_{n-1} \sigma(q) g_n^{-1}. \quad (4.6)$$

Here $\sigma(q)$ includes transport of both its source and its target; it does not mean applying σ to one of the two frames alone.

The ordinary-stabilizer surjection description is [2, Proposition 2.7.1(3)]. The following version retains the common arithmetic action and the maps induced by extension of the constant field.

Lemma 4.3. *There is a $G_{n,n-1}$ -equivariant isomorphism of v -sheaves*

$$\Psi : \mathrm{Spd} \widehat{L} \xrightarrow{\sim} \mathcal{Q}.$$

On the left the action is the actual continuous action coming from the Gross–Torii isomorphism functor, including its common arithmetic component. On the right it is (4.6). The

same assertion holds over a finite field $k_0 \supset \mathbb{F}_{p^{n(n-1)}}$, with the actual completed connected-part frame field $\widehat{L}_{k_0}$ and group

$$(\mathcal{O}_{D_n}^\times \times \mathcal{O}_{D_{n-1}}^\times) \rtimes \text{Gal}(k_0/\mathbb{F}_p).$$

These identifications commute with extension of the constant field.

Proof. The moduli problem of $\text{LT}_{n,n-1}^\infty$ remembers the height-zero special Honda marking, an integral frame of the height- $n-1$ connected part, and an integral frame of the height-one étale part. Each frame points from the fixed model to the variable object. The forgetful map

$$\text{LT}_{n,n-1}^\infty \longrightarrow \text{Spd } \widehat{L} \quad (4.7)$$

forgets the étale frame. It is the actual pro-finite étale $\mathbb{Z}_p^\times$ -torsor of such frames, by the construction of the Gross–Torii isomorphism functor and Lemma D.3.

The normalized integral comparison of Lemma D.2, followed by Lemmas D.1, D.6 and D.7, sends these data to

$$0 \longrightarrow \mathcal{O}_{X_S} \xrightarrow{j} \mathcal{E}_n \xrightarrow{q} \mathcal{E}_{n-1} \longrightarrow 0. \quad (4.8)$$

The source frame is the special frame of the shtuka; the kernel and target frames come from the slope-zero and positive-slope parts of its generic integral module, respectively. In this construction one either retains the integral generic frame, or forgets the integral lattice and reconstructs it from the rational frame by Beauville–Laszlo gluing. Lemma D.8 identifies the height-zero locus with the unit determinant fiber. Its quotient by the integral kernel-frame group is all of $\mathcal{Q}$. Since q is unchanged by changing the étale frame, the resulting equivalence descends through (4.7) to an isomorphism $\Psi : \text{Spd } \widehat{L} \xrightarrow{\sim} \mathcal{Q}$.

All these constructions commute with the two model frame groups and with their common arithmetic base change. On the meromorphic presentation, for $h = (g_n, g_{n-1}, \sigma) \in G_{n,n-1}^0$ and an étale frame change $e \in \mathbb{Q}_p^\times$, the frame convention gives

$$j \longmapsto g_n \sigma(j) e^{-1}, \quad q \longmapsto g_{n-1} \sigma(q) g_n^{-1}. \quad (4.9)$$

Rational frame changes on the integral presentation are transported by gluing; compact frame changes preserve the specified integral lattices. In particular $G_{n,n-1} \times \mathbb{Z}_p^\times$ acts on the height-zero integral tower, and its quotient action on (4.7) is the actual Gross–Torii action. Both actions change the same special marking, connected-part frame, and map from $\mathbb{F}_p$. The second formula is independent of e and is precisely (4.6).

These are continuous action maps. To see the parameter dependence explicitly, all constructions are natural on $V \times S$ for every profinite space V and continuous map $V \rightarrow G_{n,n-1}$. We use the left coefficient action $g \cdot f = (g^{-1})^* f$, as in Section 4.5. With this convention the arithmetic coefficient action is

$$\begin{aligned} \overline{\mathbb{F}}_p &\longrightarrow C(\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p), \overline{\mathbb{F}}_p), \\ a &\longmapsto (\sigma \longmapsto \sigma(a)). \end{aligned}$$

Every element of $\overline{\mathbb{F}}_p$ has finite Galois orbit, and at finite Witt length only finitely many coordinates are involved. Integral model-frame actions are the inverse limits of their finite-level actions. Applying the natural restriction, gluing and filtered comparison to these parameterized data proves equality of the action maps on $G_{n,n-1} \times \text{LT}_{n,n-1}^\infty$. Torsor descent proves the same equality on $G_{n,n-1} \times \text{Spd } \widehat{L}$, with the multiplication and identity compatibilities supplied by the same functoriality. Thus Ψ is equivariant for the actual continuous $G_{n,n-1}$ -action. All models and comparisons above are defined over a finite

field containing $\mathbb{F}_{p^{n(n-1)}}$. Repeating the same construction over such a field proves the finite-constant assertion; the gluing and frame-forgetting maps commute with extension of constants. The use of $\mathrm{Spd} \widehat{L}_{k_0}$ here is justified by Lemma C.1, independently of the surjection comparison. $\square$

We make explicit how this argument handles determinant normalization. This also distinguishes the integral étale frame torsor from the space of all rational kernel frames. Choose an isomorphism

$$\delta : \det \mathcal{E}_n \xrightarrow{\sim} \det \mathcal{E}_{n-1}$$

coming from an isomorphism of the normalized integral determinant Dieudonné modules constructed in Lemma D.8. Such a choice exists over $\overline{\mathbb{F}}_p$; it need not be fixed by the arithmetic action. For an exact sequence (4.8), define its determinant map $d_j : \det \mathcal{E}_n \rightarrow \det \mathcal{E}_{n-1}$ by

$$d_j(j(1) \wedge \tilde{v}_1 \wedge \cdots \wedge \tilde{v}_{n-1}) = v_1 \wedge \cdots \wedge v_{n-1}, \quad q(\tilde{v}_a) = v_a.$$

The order of the factors in this formula is part of our convention. In particular

$$d_{aj} = a^{-1} d_j \quad (a \in \mathbb{Q}_p^\times). \quad (4.10)$$

Every $q \in \mathcal{Q}(S)$ has a unique frame j_δ of its kernel for which $d_{j_\delta} = \delta$. Indeed, the canonical determinant isomorphism for $0 \rightarrow \ker q \rightarrow \mathcal{E}_n \rightarrow \mathcal{E}_{n-1} \rightarrow 0$ identifies

$$\ker q \simeq \det \mathcal{E}_n \otimes (\det \mathcal{E}_{n-1})^{-1},$$

and δ trivializes this line. This argument applies to families and commutes with pullback; it does not require a prior choice of a kernel frame on a cover. If $\delta' = a\delta$, then

$$j_{\delta'} = a^{-1} j_\delta. \quad (4.11)$$

In these equations scalar groups are understood as the associated locally profinite group sheaves, so the formulas also apply to sections varying over a test space.

The sheaf of all pairs (j, q) in (4.8) is a $\mathbb{Q}_p^\times$ -torsor over $\mathcal{Q}$. The height-zero infinite-level Lubin–Tate space is its subspace

$$\{(j, q) : d_j \in \mathbb{Z}_p^\times \delta\},$$

by Lemma D.8. Its fiber over q is only a $\mathbb{Z}_p^\times$ -torsor. If $d_j = a\delta$ with $a \in \mathbb{Z}_p^\times$, replacing j by aj gives the unique normalized pair. Hence

$$\frac{\{(j, q) : d_j \in \mathbb{Z}_p^\times \delta\}}{\mathbb{Z}_p^\times} \simeq \{(j, q) : d_j = \delta\} \xrightarrow{\sim} \mathcal{Q}.$$

One must therefore use $\mathbb{Z}_p^\times$ in (4.7); using all rational kernel frames instead requires the larger quotient by $\mathbb{Q}_p^\times$. These are two presentations of the same surjection sheaf, not an identification of the two frame torsors.

For $h = (g_n, g_{n-1}, \sigma) \in G_{n, n-1}^0$, define the scalar r_h by

$$\det(g_{n-1}) \circ \sigma(\delta) \circ \det(g_n)^{-1} = r_h \delta. \quad (4.12)$$

If a_σ is defined by $\sigma(\delta) = a_\sigma \delta$, this is

$$r_h = \mathrm{Nrd}_{n-1}(g_{n-1}) \mathrm{Nrd}_n(g_n)^{-1} a_\sigma.$$

All ratios here lie in $\mathbb{Q}_p^\times$ because they are automorphisms of the same rank-one isocrystal. For $h \in G_{n,n-1}$ they lie in $\mathbb{Z}_p^\times$: the two reduced norms are units, and δ and $\sigma(\delta)$ identify the same integral determinant lattices. The maps in this assertion are continuous. For example, for the standard cyclic Honda bases, the Frobenius on the determinant of the height- i isocrystal, with the normalized covariant convention of Appendix D, is $(-1)^{i-1}p^{-1}\varphi$. At every prime a determinant comparison between consecutive heights can be represented by $u = [a] - [a^p] \in W(\mathbb{F}_{p^2})^\times$ for $a \notin \mathbb{F}_p$. It satisfies $\varphi(u) = -u$. For $p = 2$, its reduction is $a + a^2 = 1$, so it is still a unit. Its arithmetic Frobenius ratio is -1 . Thus invariance of a fixed δ should not be assumed.

Starting with $d_j = \delta$, the unnormalized kernel frame $g_n\sigma(j)$ has determinant map $r_h\delta$. Equation (4.10) consequently gives the normalized action

$$j \longmapsto r_h g_n \sigma(j), \quad q \longmapsto g_{n-1} \sigma(q) g_n^{-1}. \quad (4.13)$$

Equivalently, in (4.9) the normalization is restored by the étale frame change $e = r_h^{-1}$. It is an allowed integral frame change when $h \in G_{n,n-1}$. The ratios satisfy $r_{h_1 h_2} = r_{h_1} \sigma_1(r_{h_2})$; this follows directly by applying the two frame changes to δ . Thus (4.13) is a coherent continuous action. The extra factor affects only the kernel frame and disappears on forgetting that frame. This recovers (4.6) without making an arithmetic invariance assumption on δ .

For an ordinary frame change $(g_n, g_{n-1}, 1)$, let e_q be the class of the extension with δ -normalized kernel. The action is

$$\begin{aligned} j &\longmapsto \frac{\mathrm{Nrd}_{n-1}(g_{n-1})}{\mathrm{Nrd}_n(g_n)} g_n j, \\ e_q &\longmapsto \frac{\mathrm{Nrd}_n(g_n)}{\mathrm{Nrd}_{n-1}(g_{n-1})} (g_{n-1}^{-1})^* e_q. \end{aligned}$$

Indeed, g_n is an isomorphism of middle bundles, the right end changes by g_{n-1} , and the normalized map on the left end is multiplication by $\mathrm{Nrd}_n(g_n)/\mathrm{Nrd}_{n-1}(g_{n-1})$. Naturality of extension classes gives the displayed pullback and pushout formula. For a general arithmetic element $h = (g_n, g_{n-1}, \sigma)$, the same argument gives

$$e_q \longmapsto r_h^{-1} (g_{n-1}^{-1})^* \sigma(e_q).$$

The arithmetic descent of the extension space therefore retains the determinant comparison's cocycle. In particular, the residual source norm a acts by the pure dilation $e_q \mapsto a e_q$. The normalization changes only the kernel frame; it changes neither stabilizer group, its projection, nor the source cohomology class.

Proposition 4.4. *There is a morphism*

$$F : \mathcal{C} \longrightarrow \mathcal{Y} = [\mathrm{Spd} \hat{L}/G_{n,n-1}]$$

whose pullback of the tautological $G_{n,n-1}$ -torsor is P . Projection of this torsor to G_n and G_{n-1} gives P_n and P_{n-1} , respectively. In particular, the source classifying map of F , followed by $BG_{n,n-1} \rightarrow BG_n$, is p_n .

Proof. On P , the constants agree and the two middle bundles have frames

$$\alpha : \mathcal{O}(1/n) \xrightarrow{\sim} \mathcal{V}_n, \quad \beta : \mathcal{O}(1/(n-1)) \xrightarrow{\sim} \mathcal{V}_{n-1}.$$

The exact sequence (4.3) gives the surjection

$$q_P = \beta^{-1} q_V \alpha : \mathcal{O}(1/n) \twoheadrightarrow \mathcal{O}(1/(n-1)).$$

Let $\Psi : \mathrm{Spd} \widehat{L} \xrightarrow{\sim} \mathcal{Q}$ be the actual arithmetic-equivariant comparison of Lemma 4.3, and put $\tilde{F} = \Psi^{-1} \circ q_P$.

Here the two frames are the specified frames supplied by P_n and P_{n-1} . With the left frame convention, a simultaneous semilinear model change $A_i = g_i \sigma$ changes a frame by right composition with A_i^{-1} . Consequently

$$q_P \longmapsto A_{n-1} q_P A_n^{-1} = g_{n-1} \sigma(q_P) g_n^{-1}.$$

The equality includes transport of both reference bundles by their common arithmetic descent datum. It holds on $G_{n,n-1} \times P$: the constructions commute with base change, including profinite parameter spaces. Lemma 4.3 identifies this action with the actual action on $\mathrm{Spd} \widehat{L}$. Thus $\tilde{F}$ is equivariant.

Taking its quotient produces F , with a Cartesian square of v -stacks

$$\begin{array}{ccc} P & \xrightarrow{\tilde{F}} & \mathrm{Spd} \widehat{L} \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{F} & [\mathrm{Spd} \widehat{L} / G_{n,n-1}]. \end{array}$$

The square is Cartesian by the construction of a map into a quotient stack from a torsor and an equivariant map. Its tautological pullback is therefore the original torsor P . For $i = n, n-1$, extension of structure group gives an isomorphism

$$G_i \times^{G_{n,n-1}} P \xrightarrow{\sim} P_i, \quad [g, (\xi_n, \xi_{n-1})] \longmapsto g \xi_i.$$

It is well defined by the contracted-product relation and is an isomorphism after trivializing the torsors, hence globally. This proves the two projection assertions and identifies the source classifying map with p_n . $\square$

Proposition 2.9 identifies the coefficient transfer with constant cohomology on $\mathcal{Y}$. Pulling (2.21) back along F gives

$$F^* f_{\mathrm{target}}^* \kappa_{\Gamma \times G_{n-1}} (\tau(\inf_{G_n}^{G_{n,n-1}} \vartheta_n)) = p_n^* \kappa_{G_n}(\vartheta_n) \quad \text{in } H_v^3(\mathcal{C}, \mathbb{Z}_p)[1/p]. \quad (4.14)$$

4.4 The source Cartan class on the test family

For a complex $\mathcal{F}$ of abelian sheaves on $X_{\mathrm{pro\acute{e}t}}$, we follow [3, §3.6] and write $R\Gamma_{\mathrm{cond}}(X_{\mathrm{pro\acute{e}t}}, \mathcal{F})$ for its geometric condensed cohomology. It takes values in $D(\mathrm{Cond}(\mathbf{Ab}))$, the derived category of condensed abelian groups; evaluation at the one-point profinite set gives ordinary derived cohomology.

Write unit_* for the map

$$H_v^3(\mathcal{C}, \mathbb{Z}_p)[1/p] \xrightarrow{\sim} H_{\mathrm{pro\acute{e}t}}^3(\mathcal{C}, \mathbb{Z}_p)[1/p] \longrightarrow H_{\mathrm{pro\acute{e}t}}^3(\mathcal{C}, \widehat{\mathcal{O}}^+)[1/p].$$

The first arrow is the constant-site comparison of Appendix E.5; the second is the coefficient unit.

We apply the tower and integral cohomology comparisons of [3, Theorems 3.9.1 and 3.9.4] to the source Cartan line. The next lemma records its restriction to the last SL_2 block and identifies it with the unit image on the specified torsor P_n .

Lemma 4.5. *The actual source torsor satisfies*

$$\mathrm{unit}_* p_n^* \kappa_{G_n}(\vartheta_n) \neq 0 \quad \text{in } H_{\mathrm{pro\acute{e}t}}^3(\mathcal{C}, \widehat{\mathcal{O}}^+)[1/p].$$

Proof. The coefficient-unit image of ϑ_n in $H_{\text{cts}}^3(G_n, R_n)[1/p]$ is nonzero by Lemma 3.4. We first transport this class to the Drinfeld quotient, then restrict it to the last SL_2 block.

The Lubin–Tate quotient. Lemma E.1 gives the G_n -equivariant comparison

$$\alpha_{\text{LT}} : R_n[\epsilon_{\text{LT}}] \longrightarrow R\Gamma_{\text{cond}}((\text{LT}_{n,\eta})_{\text{proét}}, \hat{\mathcal{O}}^+),$$

where $|\epsilon_{\text{LT}}| = 1$ and ϵ_{LT} has trivial action. Its degree-zero restriction is the coefficient map. The cofiber is in $D^{\geq 1}$, and its first two cohomology objects are killed by powers of p . Lemma E.4 therefore gives

$$H^2((\text{cofib } \alpha_{\text{LT}})^{hG_n})[1/p] = 0.$$

The long exact sequence shows that α_{LT} induces an injection in degree three after inverting p . The inclusion $R_n \rightarrow R_n[\epsilon_{\text{LT}}]$ has its equivariant augmentation as a retraction. Thus the constant Cartan class has nonzero image on $[\text{LT}_{n,\eta}^\circ / G_n]$.

The Drinfeld quotient. Transport this nonzero image by the actual tower identification (4.4) in rank n . The cohomological form of the tower identification is [3, Theorem 3.9.1 and equation (3.9.3)]. By the moduli description used to define P_n , the transported class is the completed-structure-sheaf image of the universal source G_n -torsor.

The Drinfeld comparison of Lemma E.1 is

$$\alpha_{\text{Dr}} : \mathbb{Z}_p[\epsilon_{\text{Dr}}] \longrightarrow R\Gamma_{\text{cond}}((\mathbb{H}^{n-1})_{\text{proét}}, \hat{\mathcal{O}}^+).$$

Both source summands have trivial $\text{GL}_n(\mathbb{Z}_p)$ -action. Its cofiber is in $D^{\geq 1}$, with its first three cohomology objects killed by powers of p . Lemma E.4 kills degree two and degree three of the cofiber after $\text{GL}_n(\mathbb{Z}_p)$ -invariants and inversion of p . Hence the comparison induces an isomorphism in degree three.

Proposition 3.8.1 of [3] gives

$$\begin{aligned} H_{\text{cts}}^2(\text{GL}_n(\mathbb{Z}_p), \mathbb{Z}_p)[1/p] &= 0, \\ H_{\text{cts}}^3(\text{GL}_n(\mathbb{Z}_p), \mathbb{Z}_p)[1/p] &= \mathbb{Q}_p y_3, \end{aligned}$$

where y_3 is a nonzero matrix Cartan class. The Lie comparison descends from an open analytic subgroup by finite averaging over $\mathbb{Q}_p$, so it applies even when $\text{GL}_n(\mathbb{Z}_p)$ has p -torsion. The cofiber argument above uses the finite low-degree filtration of Lemma E.4, which requires no finite integral cohomological dimension. The degree-three contribution of ϵ_{Dr} is consequently zero. The transported source class is a nonzero element of the line obtained from y_3 by the coefficient unit.

The last SL_2 block. Restrict α_{Dr} to $\text{SL}_2(\mathbb{Z}_p)$. Its cofiber retains the same first two cohomology bounds, so Lemma E.4 again gives an injection in degree three after invariants and inversion of p . The augmentation of $\mathbb{Z}_p[\epsilon_{\text{Dr}}]$ shows that the coefficient-unit map is injective on $H_{\text{cts}}^3(\text{SL}_2(\mathbb{Z}_p), \mathbb{Z}_p)[1/p]$.

Under the natural Lazard comparison on sufficiently small uniform subgroups, restriction of y_3 is represented on the last block by

$$(X, Y, Z) \longmapsto \text{Tr}(X[Y, Z]), \quad X, Y, Z \in \mathfrak{sl}_2(\mathbb{Q}_p).$$

For the basis

$$\mathbf{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

this form has value $\text{Tr}(\mathbf{H}[\mathbf{E}, \mathbf{F}]) = 2$. Every alternating two-cochain ω satisfies

$$(d\omega)(\mathbf{H}, \mathbf{E}, \mathbf{F}) = -2\omega(\mathbf{E}, \mathbf{F}) - 2\omega(\mathbf{F}, \mathbf{E}) - \omega(\mathbf{H}, \mathbf{H}) = 0.$$

Thus the restricted Cartan cocycle is not a coboundary. Naturality of Lazard's comparison [3, §3.8 and the proof of Proposition 3.8.1] shows that $\text{res}_{\text{SL}_2(\mathbb{Z}_p)} y_3 \neq 0$. The coefficient-unit image of the transported class therefore remains nonzero on $\mathcal{C}$.

Finally, the common tower and the constant-site comparison identify this restriction with $\text{unit}_* p_n^* \kappa_{G_n}(\vartheta_n)$. The torsor used here is P_n , defined by the same moduli equivalence, and the comparison preserves its constant unit by Appendix E.5. $\square$

4.5 Target coefficients on the test family

We construct coefficient evaluation directly on the torsor nerve. For a profinite set U , derived evaluation of condensed cohomology gives

$$R\Gamma_{\text{cond}}(X_{\text{proét}}, \mathcal{F})(U) \simeq R\Gamma((X \times U)_{\text{proét}}, \mathcal{F}).$$

The right side denotes ordinary derived sections; on $X \times U$, and on torsors over X below, $\mathcal{F}$ denotes its pullback. For a stack, these expressions use the continuous quotient-nerve convention of Appendix E.5. We use the untilted completed integral structure sheaf $\hat{\mathcal{O}}^+$ on all the spaces below. Let $\hat{\mathcal{O}}_{\text{cond}}^+$ denote the condensed coefficient sheaf obtained from its topological sections, as in [3, §3.5]. The comparison of [3, Lemma 3.6.1 and (3.6.2)] identifies

$$R\Gamma_{\text{cond}}(X_{\text{proét}}, \hat{\mathcal{O}}^+) \simeq R\Gamma(X_{\text{proét}}, \hat{\mathcal{O}}_{\text{cond}}^+).$$

In particular, on an affinoid perfectoid chart $V^\sharp = \text{Spa}(R, R^+)$,

$$\Gamma(V \times U, \hat{\mathcal{O}}^+) = C(U, R^+).$$

We use this comparison only for the completed structure sheaf.

Let G be profinite and let $P \rightarrow X$ be a left pro-étale G -torsor. On sections of a sheaf pulled back from X , use the left action $g \cdot f = (g^{-1})^* f$. The torsor-nerve comparison is the natural equivalence

$$R\Gamma_{\text{cond}}(X_{\text{proét}}, \mathcal{F}) \xrightarrow{\sim} R\Gamma_{\text{cts}}(G, R\Gamma_{\text{cond}}(P_{\text{proét}}, \mathcal{F})). \quad (4.15)$$

Indeed, $P^{\times x(q+1)} \simeq P \times G^q$, and the derived sections on these terms form the continuous action-cochain diagram. Applying this description also after product with every profinite U proves the internal comparison. The torsor-descent formula is [4, Proposition 4.1.3]. Applying its action-nerve proof after product with each profinite U gives the internal comparison above; for $\hat{\mathcal{O}}^+$, see also [3, Proposition 3.6.3]. The same argument applies on a presentation of a stack and its nerve, since both descent operations are limits.

Suppose A is a solid $\mathbb{Z}_p$ -algebra with continuous G -action and that the torsor carries a continuous equivariant coefficient evaluation. Concretely, this means compatible maps

$$A(U) \longrightarrow \Gamma(V \times U, \hat{\mathcal{O}}^+)$$

for every profinite U and every affinoid perfectoid chart $V \rightarrow P$, natural under restriction of charts and profinite parameters. We require them to be unital and compatible with the action on $G^q \times P \times U$ for every q . With the left-action convention above, this compatibility reads $\text{ev}_{g^v}(ga) = \text{ev}_v(a)$ on a trivializing chart. Equivalently, $\varepsilon_{P,A}(ga) = (g^{-1})^* \varepsilon_{P,A}(a)$. They define a G -equivariant coaugmentation

$$\varepsilon_{P,A} : A \longrightarrow R\Gamma_{\text{cond}}(P_{\text{proét}}, \hat{\mathcal{O}}^+).$$

Explicitly, for each profinite U the chart maps glue to $A(U) \rightarrow \Gamma(P \times U, \hat{\mathcal{O}}^+)$. Composing with the canonical maps to derived sections gives the displayed coaugmentation. Thus this construction does not require a choice of a geometric associated sheaf for A .

Taking continuous fixed points and inverting (4.15) defines

$$e_{P,A} : R\Gamma_{\text{cts}}(G, A) \longrightarrow R\Gamma_{\text{cond}}(X_{\text{proét}}, \widehat{\mathcal{O}}^+). \quad (4.16)$$

All maps here are integral maps of derived condensed complexes. For the solid structure-sheaf complexes, this is equivalently the solid construction furnished by the cited comparisons. Taking cohomology and evaluating at the point gives maps to ordinary pro-étale cohomology.

Lemma 4.6 (Naturality and the constant unit). *The maps $e_{P,A}$ are natural under pullback of torsors and compatible equivariant coefficient maps. Let $\phi : G \rightarrow H$ be a continuous homomorphism, $Q = H \times^G P$, and let A be an H -algebra with a coefficient evaluation on Q . Equip P with its pullback evaluation. Then*

$$e_{P, \phi^* A} \circ \phi^* = e_{Q,A}. \quad (4.17)$$

Here ϕ^* on cohomology means inflation or restriction along the indicated homomorphism, without requiring it to be injective or surjective.

Write

$$\mathbf{Z}_{p,X} = R\varprojlim_m (\mathbb{Z}/p^m)_X$$

for the geometric constant coefficient complex, and let $u_X : \mathbf{Z}_{p,X} \rightarrow \widehat{\mathcal{O}}^+$ be its unit. Let

$$\kappa_P^{\text{proét}} : R\Gamma_{\text{cts}}(G, \mathbb{Z}_p) \longrightarrow R\Gamma_{\text{cond}}(X_{\text{proét}}, \mathbf{Z}_{p,X})$$

be the constant-unit map on the actual torsor nerve. For a unital coefficient evaluation,

$$e_{P,A} \circ R\Gamma_{\text{cts}}(G, \text{unit}_A) = R\Gamma_{\text{cond}}(X_{\text{proét}}, u_X) \circ \kappa_P^{\text{proét}}. \quad (4.18)$$

Under the constant-site comparison, the latter torsor map is the pullback of κ_G along the classifying map of P .

Proof. Pullback and equivariant coefficient maps give maps between the parameterized coaugmentations $\varepsilon_{P,A}$ and between their action-cochain diagrams. Totalizing these maps proves the first naturality assertion. For change of group, the equivariant map $P \rightarrow Q$ and $G^q \rightarrow H^q$ give a map of torsor nerves over X . The pulled-back evaluations agree term by term. Applying descent gives (4.17). This argument includes all faces, degeneracies, and their coherences, and remains valid with an additional parameter U .

For the unit assertion, first fix m and use finite constant coefficients. A section of $C(G^q \times U, \mathbb{Z}/p^m)$ gives its constant-function section on $P \times G^q \times U$. Composing with the coefficient unit gives the same map to

$$R\Gamma((P \times G^q \times U)_{\text{proét}}, \widehat{\mathcal{O}}^+ \otimes^L \mathbb{Z}/p^m)$$

as the reduction of $\varepsilon_{P,A} \circ \text{unit}_A$. This is checked on an affinoid perfectoid chart: both maps send a locally constant scalar to that scalar in the untilted coefficient ring modulo p^m . The maps commute with length reduction and with every map of the torsor nerve.

Now take the derived inverse limit over m . The source constant condensed module is $\mathbb{Z}_p \simeq R\varprojlim_m \mathbb{Z}/p^m$; this can also be checked on profinite sets, where the transition maps on continuous finite-valued functions are surjective. The geometric constant complex is the displayed derived inverse limit by definition. The target is the derived p -complete integral structure sheaf used in the coefficient comparison, so

$$\widehat{\mathcal{O}}^+ \simeq R\varprojlim_m (\widehat{\mathcal{O}}^+ \otimes^L \mathbb{Z}/p^m).$$

On the untilted perfectoid charts its sections are p -torsion-free and p -adically complete; this is the completed structure-sheaf construction of [3, §3.6] and Appendix F.1. Derived sections, continuous fixed points, and totalization preserve limits. The finite-level equalities consequently give (4.18). No inverse-image functor is moved past this inverse limit.

Finally, κ_G was defined by the same constant-unit maps on the action groupoid. Pulling that groupoid back along the classifying map of P gives this torsor nerve. Appendix E.5 compares its finite constant cochain diagrams, with all profinite parameters, before taking the derived inverse limit. It therefore identifies the two constant torsor maps and their structure-sheaf units. $\square$

Apply this construction to $X = \mathcal{C}$, $G = G_{n-1}$, $P = P_{n-1}$, and $A = R_{n-1}$. The equivariant map $\tilde{m}_{n-1} : P_{n-1} \rightarrow \mathrm{LT}_{n-1,\eta}^\diamond$ supplies the required coefficient evaluation. For an affinoid perfectoid chart $V^\sharp = \mathrm{Spa}(R, R^+)$ over this torsor, the universal deformation defines the unital map

$$R_{n-1}(\ast) = W(\overline{\mathbb{F}}_p)[[u_1, \dots, u_{n-2}]] \longrightarrow R^+.$$

The parameters of its maximal ideal map to topologically nilpotent elements, so the map is continuous. Applying it to continuous functions on a profinite U gives $R_{n-1}(U) \rightarrow C(U, R^+)$. Indeed, the pointwise deformation construction identifies $R_{n-1}(U)$ with the continuous functions into its adic coefficient ring; this also follows by taking its finite adic quotients, whose residue coefficients are discrete. These maps commute with restriction to charts and with the actual action on $G_{n-1}^q \times P_{n-1} \times U$. They are the universal coefficient evaluations used in [3, §3.6 and Theorem 3.9.4].

We therefore have the direct evaluation

$$e_{P_{n-1}, R_{n-1}} : R\Gamma_{\mathrm{cts}}(G_{n-1}, R_{n-1}) \longrightarrow R\Gamma_{\mathrm{cond}}(\mathcal{C}_{\mathrm{proét}}, \hat{\mathcal{O}}^+). \quad (4.19)$$

Lemma 4.6 gives the commutative square of derived complexes below. The bottom arrow is induced by the coefficient unit $u_{\mathcal{C}}$ and is denoted $u_{\mathcal{C}, \ast}$.

$$\begin{array}{ccc} R\Gamma_{\mathrm{cts}}(G_{n-1}, \mathbb{Z}_p) & \xrightarrow{\mathrm{unit}_\ast} & R\Gamma_{\mathrm{cts}}(G_{n-1}, R_{n-1}) \\ \kappa_{P_{n-1}}^{\mathrm{proét}} \downarrow & & \downarrow e_{P_{n-1}, R_{n-1}} \\ R\Gamma_{\mathrm{cond}}(\mathcal{C}_{\mathrm{proét}}, \mathbf{Z}_p, \mathcal{C}) & \xrightarrow{u_{\mathcal{C}, \ast}} & R\Gamma_{\mathrm{cond}}(\mathcal{C}_{\mathrm{proét}}, \hat{\mathcal{O}}^+). \end{array} \quad (4.20)$$

The same construction applies to every change of structure group of the torsors on $\mathcal{C}$, using the pulled-back evaluation.

Proof of Proposition 4.1. Use the quotient $\Gamma \times G_{n-1}$ and $\mathrm{pr} : \Gamma \times G_{n-1} \rightarrow G_{n-1}$, and write

$$z = \tau(\inf_{G_n}^{G_{n,n-1}} \vartheta_n) \in H_{\mathrm{cts}}^3(\Gamma \times G_{n-1}, \mathbb{Z}_p)[1/p].$$

By definition of η and (4.1),

$$\mathrm{unit}_{R_{n-1}, \ast} z = \eta = \mathrm{pr}^\ast b.$$

Let T be the $\Gamma \times G_{n-1}$ -torsor obtained from P by extension of structure group along $G_{n,n-1} \rightarrow \Gamma \times G_{n-1}$. It is the pullback along F of the target torsor of $\mathcal{Y}$, and extending T along pr gives P_{n-1} . Pull the evaluation on P_{n-1} back along the canonical map $T \rightarrow P_{n-1}$. Since Γ acts trivially on R_{n-1} , this is the required $\Gamma \times G_{n-1}$ -equivariant evaluation on T .

Change-of-group naturality and the unit identity of Lemma 4.6 give

$$\begin{aligned} e_{P_{n-1}, R_{n-1}, *}(b) &= e_{T, R_{n-1}, *}(\mathrm{pr}^* b) \\ &= e_{T, R_{n-1}, *}(\mathrm{unit}_{R_{n-1}, *}(z)) \\ &= u_{\mathcal{C}, *} \kappa_T^{\mathrm{pro\acute{e}t}}(z). \end{aligned}$$

The torsor defining the last constant class is exactly the pullback of the target torsor along F . Thus (4.14), followed by the constant-site comparison, identifies its structure-sheaf image with

$$u_{\mathcal{C}, *} \kappa_T^{\mathrm{pro\acute{e}t}}(z) = \mathrm{unit}_* p_n^* \kappa_{G_n}(\vartheta_n).$$

Here unit_* is the map fixed before Lemma 4.5. The right side is nonzero by that lemma. Hence $b \neq 0$.

The maps and their identities were constructed integrally on the actual parameterized torsor nerves. Degree-three cohomology, evaluation at the point, and inversion of p are applied only afterwards. $\square$

5 The non-splitting obstruction

Proof of Theorem 1.1. Let ϑ_n be the constant Cartan class of Lemma 3.4, and let x_n be its coefficient-unit image under the natural comparison

$$H_{\mathrm{cts}}^3(G_n, R_n)_{\mathbb{Q}} \simeq \pi_{-3} L_{K(n)} S_p \otimes \mathbb{Q}.$$

Inflate ϑ_n along $G_{n, n-1} \rightarrow G_n$ and apply the actual transfer of Proposition 2.2. As in (4.1), its coefficient-unit image is

$$\eta = \mathrm{pr}^* b, \quad b \in H_{\mathrm{cts}}^3(G_{n-1}, R_{n-1})_{\mathbb{Q}}.$$

Proposition 4.1 gives $b \neq 0$. Let

$$y \in \pi_{-3} L_{K(n-1)} S_p \otimes \mathbb{Q}$$

be the class corresponding to b under Lemma 3.2. Then $y \neq 0$.

The coefficient square of Proposition 2.2 identifies the image of the source class in $H_{\mathrm{cts}}^3(G_{n, n-1}, \pi_0 B)_{\mathbb{Q}}$ with the image of b under the actual target coefficient map $R_{n-1} \rightarrow \pi_0 B$, inflated along $G_{n, n-1} \rightarrow G_{n-1}$. Naturality of Lemma 3.2 therefore identifies these two classes in $\pi_{-3} B^{hG_{n, n-1}}(*) \otimes \mathbb{Q}$.

Under the source descent equivalence of Lemma 2.1, the map

$$E_n^{hG_n}(*) \longrightarrow B^{hG_{n, n-1}}(*)$$

is ℓ . The target map

$$E_{n-1}^{hG_{n-1}}(*) \longrightarrow B^{hG_{n, n-1}}(*) \simeq L_{K(n-1)} L_{K(n)} S_p$$

is j : it preserves the sphere unit, and its codomain is $K(n-1)$ -local. Indeed, precomposition with $S_p \rightarrow L_{K(n-1)} S_p$ identifies maps into that codomain, so the unit determines this map. Consequently

$$\ell_*(x_n) = j_*(y), \quad y \neq 0.$$

Proposition 1.2 proves the theorem. $\square$

A Descent and inversion for solid spectra

Lemma 3.2 uses the following comparison to pass from continuous coefficient cohomology to rational homotopy. The Morava descent step applies [5, Proposition 2.12] to the descendable solid Galois extension of [2, Corollary 4.6.8 and Proposition 4.6.9]. For the mixed spectrum, the same argument uses the target coefficient acyclicity of Lemma A.6.

Lemma A.1. *For the three pairs in (3.1), the canonical comparison $(X^{hG})[1/p] \rightarrow (X[1/p])^{hG}$ is an equivalence. These comparisons are natural for the actual source and target maps into B , with inflation along the two projections.*

The cotensor and finite Postnikov arguments below also support the central-weight reduction in Lemma 3.2. They are formal consequences of the solid group-ring framework [6, Lemmas 2.11(3), 2.13, and 2.14]. The cohomological dimension bound is used only for the central group $\mathbb{Z}_p$.

Lemma A.2. *Let S be a profinite set and X a solid spectrum whose homotopy sheaves are solid $\mathbb{Z}_p$ -modules. Profinite cotensor is t -exact on these spectra, and the canonical map*

$$C(S, X)[1/p] \longrightarrow C(S, X[1/p])$$

is an equivalence, natural in S and X . Their Postnikov towers are complete.

Proof. For every profinite T , the module $\mathbb{Z}_p[S \times T]^\square$ is compact projective [6, Lemma 2.11(3)]. Consequently cotensor by S is exact on the solid module heart, including internally, and commutes there with the filtered telescope that inverts p .

These assertions pass to spectra by Postnikov truncation. For a spectrum bounded above by m , output degree j only involves the finite interval of homotopy modules from j to m : the lower remainder is coconnective below j , and cotensor preserves coconnective spectra. On each of these modules, profinite sections have cohomological dimension zero by the projectivity just noted. Thus cotensor commutes with homotopy sheaves on each bounded-above truncation.

Evaluation on extremally disconnected profinite sets preserves truncations and limits. Ordinary spectra have complete Postnikov towers, so the solid spectra in question also do. The towers after cotensor stabilize on each fixed homotopy group and its successor; their Milnor $\varprojlim^1$ terms vanish. It follows that cotensor commutes with homotopy sheaves for X itself. For an extremally disconnected T , the relevant formula is

$$\pi_j C(S, X)(T) = \mathrm{Hom}_{\mathrm{Solid}_{\mathbb{Z}_p}}(\mathbb{Z}_p[S \times T]^\square, \pi_j X).$$

Compactness makes this formula commute with inversion of p . Homotopy sheaves detect equivalences, proving the claimed map. $\square$

Lemma A.3. *Let G be a profinite group and let $D \in D(\mathrm{Solid}_{\mathbb{Z}_p})$ be cohomologically bounded below, with continuous G -action. Then*

$$R\Gamma_{\mathrm{cts}}(G, D)[1/p] \xrightarrow{\sim} R\Gamma_{\mathrm{cts}}(G, D[1/p]).$$

The comparison is internal and natural in the group, coefficients, and profinite parameters.

Proof. For a single module, each bar term is profinite cotensor. Lemma A.2 makes it commute with inversion of p . Normalization uses finite kernels, and inversion of p is exact, so the normalized cochain complex gives the asserted comparison.

Suppose $D \in D^{\geq b}$. Derived continuous invariants preserve cohomological lower bounds, so output degree r only depends on the finite truncation $\tau_{[b,r]}D$. Its finite Postnikov filtration reduces the assertion in that degree to the preceding module calculation. Equivalently, the total-degree- r bar calculation only uses $b \leq v \leq r$ and $0 \leq s = r - v \leq r - b$. This proves the comparison in every degree, naturally. $\square$

Lemma A.4. *The group $C \simeq \mathbb{Z}_p$ has internal cohomological dimension at most one for solid $\mathbb{Z}_p$ -module coefficients with continuous C -action.*

Proof. For a topological generator c , put $\Lambda = \mathbb{Z}_p[[C]] \simeq \mathbb{Z}_p[[T]]$, with $T = c - 1$. Its augmentation resolution is

$$0 \longrightarrow \Lambda \xrightarrow{T} \Lambda \longrightarrow \mathbb{Z}_p \longrightarrow 0.$$

This is an exact sequence of profinite modules. Condensation preserves exactness, and $\underline{\Lambda} = \mathbb{Z}_p[C]^\square$ [6, Lemma 2.13]. The equivalence between solid completed-group-ring modules and continuous solid actions is [6, Lemma 2.14]. Both free terms are internally projective: Hom from a rank-one free group-ring module is the underlying-module functor. The resolution therefore computes internal derived invariants in degrees zero and one. $\square$

Lemma A.5 (Finite Postnikov window). *Let G have internal cohomological dimension at most d for solid $\mathbb{Z}_p$ -module coefficients. Let X be a Postnikov-complete solid spectrum with continuous G -action and solid $\mathbb{Z}_p$ -linear homotopy sheaves. For every $j \in \mathbb{Z}$, there is a natural isomorphism*

$$\pi_j(X^{hG}) \simeq \pi_j((\tau_{[j,j+d]}X)^{hG}).$$

Proof. The fiber between Postnikov stages m and $m-1$, after fixed points, is $(\Sigma^m H\pi_m X)^{hG}$. Its homotopy sheaves lie in $[m-d, m]$. Thus the fixed-point tower stabilizes on degree j from stage $j+d$, and on degree $j+1$ from stage $j+d+1$. Completeness and preservation of limits by fixed points give

$$X^{hG} \simeq \varprojlim_m (\tau_{\leq m} X)^{hG}.$$

After evaluation on an extremally disconnected profinite parameter, stabilization in these two degrees kills the Milnor $\varprojlim^1$ term. Hence degree j is computed at stage $j+d$. Finally, in the triangle

$$\tau_{[j,j+d]}X \longrightarrow \tau_{\leq j+d}X \longrightarrow \tau_{\leq j-1}X,$$

the last term remains coconnective below j after fixed points. Its homotopy in degrees j and $j+1$ is zero, which proves the assertion, also internally. $\square$

The residue input is [2, Proposition 4.6.16] for a uniform pro- p subgroup. We pass to the full target stabilizer by finite Galois descent and lift through the adic filtration, retaining all periodicity twists.

Lemma A.6. *The actual source unit induces, for every $v \in \mathbb{Z}$,*

$$\pi_v L_{K(n-1)} E_n \xrightarrow{\sim} R\Gamma_{\text{cts}}(\mathcal{O}_{D_{n-1}}^\times, \pi_v B).$$

The left side is concentrated in cohomological degree zero. These equivalences preserve the residual G_n -action.

Proof. We first establish residue descent for the full group $\mathcal{O}_{D_{n-1}}^\times$. The Gross-Torii extension is the smooth $\mathcal{O}_{D_{n-1}}^\times$ -Galois extension

$$\overline{\mathbb{F}}_p((u_{n-1})) \longrightarrow L$$

of [2, Example 4.6.3]. Here L has the ind-field coefficient structure (2.2): it is the filtered colimit of its finite Galois subextensions with their valuation topologies. For sufficiently large r , the subgroup $U = 1 + p^r \mathcal{O}_{D_{n-1}}$ is open, normal, and uniform. Indeed, logarithm identifies it with the lattice $p^r \mathcal{O}_{D_{n-1}}$; commutators lie in $p^{2r} \mathcal{O}_{D_{n-1}}$, and taking p -th powers corresponds to multiplication by p on this lattice. For $p = 2$ choose $r \geq 2$. The subgroup is also stable under arithmetic conjugation. The extension L/L^U , with the same ind-field structure, is smooth U -Galois. Thus [2, Proposition 4.6.16] gives the actual augmentation

$$L^U \xrightarrow{\sim} R\Gamma_{\text{cts}}(U, L).$$

The extension $L^U/\overline{\mathbb{F}}_p((u_{n-1}))$ is finite Galois with group $\mathcal{O}_{D_{n-1}}^\times/U$. By the normal basis theorem its additive module is isomorphic to

$$C(\mathcal{O}_{D_{n-1}}^\times/U, \overline{\mathbb{F}}_p((u_{n-1}))),$$

with the regular action. This is also a topological isomorphism: coordinates in any basis of a finite extension of complete valued fields give a homeomorphism. It therefore remains an isomorphism with every profinite parameter, and identifies the additive module with a coinduced module for the finite group $\mathcal{O}_{D_{n-1}}^\times/U$. The coinduced bar complex is contractible in positive degrees, including when p divides $|\mathcal{O}_{D_{n-1}}^\times/U|$. In degree zero the canonical augmentation identifies the invariants with $\overline{\mathbb{F}}_p((u_{n-1}))$. Iterated fixed points consequently give

$$\overline{\mathbb{F}}_p((u_{n-1})) \xrightarrow{\sim} R\Gamma_{\text{cts}}(\mathcal{O}_{D_{n-1}}^\times, L).$$

The normal basis proves acyclicity; the displayed equivalence is induced by the original field unit. Since that unit is equivariant and U is arithmetic-stable, the equivalence retains the residual G_n -action. Cotensor by a profinite set preserves the equivalence and commutes with its fixed-point totalizations.

The source witness is the complete local ring in (2.7). Its maximal ideal is generated by $p, u_1, \dots, u_{n-2}$, and its residue field is $\overline{\mathbb{F}}_p((u_{n-1}))$. The target group fixes this ring. For every profinite parameter T , the degree-zero augmentation has flat modules over this ring that are complete for its maximal ideal, and all structure maps are linear over it. These are the pointwise deformation and cotensor properties of [2, Proposition 4.5.5 and Remark 4.5.6], with flatness and the ideal identification supplied by [2, Proposition 4.2.4 and Lemma 4.2.12]; see also Lemma B.1. Lemma B.2 lifts the residue augmentation through this adic filtration and proves the claimed degree-zero equivalence.

The image of a periodicity unit of E_n is invertible in $L_{K(n-1)}E_n$ and B , and is fixed by $\mathcal{O}_{D_{n-1}}^\times$. Every even homotopy module is the corresponding source periodicity twist of degree zero; odd homotopy modules vanish. This proves the assertion for every v . All augmentations used in the adic lift and periodicity twists come from the actual source unit, so the resulting equivalences retain their residual G_n -action. $\square$

Proof of Lemma A.1. Morava descent. By [2, Corollary 4.6.8 and Proposition 4.6.9], $L_{K(i)}S_p \rightarrow E_i$ is descendable, with augmented Amitsur complex $L_{K(i)}S_p \rightarrow C(G_i^\bullet, E_i)$. A descendable Amitsur complex remains a limit diagram after any exact functor [5, Proposition 2.12]. Apply this first to the exact inclusion of solid $K(i)$ -local spectra into solid spectra, followed by ordinary inversion of p . Lemma A.2 identifies the resulting terms with $C(G_i^\bullet, E_i[1/p])$. This proves the canonical comparison for E_i .

For the localized source, apply the same proposition to the height- n augmented complex, first composing the inclusion with the plain exact functor $L_{K(n-1)}$, and then with ordinary

inversion of p . The profinite-cotensor identifications [2, Remark 4.5.6 and Propositions 4.5.9–4.5.10], as used in [2, Corollary 4.6.10], identify its terms as

$$L_{K(n-1)}C(G_n^s, E_n) \simeq C(G_n^s, L_{K(n-1)}E_n).$$

The proposition before inversion identifies the integral augmentation with the fixed points of this complex; after inversion it identifies its inverted augmentation with the fixed points of the inverted complex. Hence

$$((L_{K(n-1)}E_n)^{hG_n})[1/p] \xrightarrow{\sim} ((L_{K(n-1)}E_n)[1/p])^{hG_n}. \quad (\text{A.1})$$

Both applications use the underlying solid spectra and the original augmented diagrams. Their equivalences are the canonical inversion maps, so they remain compatible with equivariant coefficient maps and group inflation.

Target coefficient descent. Lemma A.6 and Lemma A.3 give

$$(\pi_v L_{K(n-1)}E_n)[1/p] \simeq R\Gamma_{\text{cts}}(\mathcal{O}_{D_{n-1}}^\times, \pi_v B[1/p]) \quad (v \in \mathbb{Z}).$$

To pass from these modules to spectra, truncate both already inverted spectra above a fixed degree m . The bar terms are bounded above by m , and the fiber between partial totalizations s and $s-1$ is bounded above by $m-s$. Thus, on every fixed degree j and its successor, the partial-totalization tower eventually stabilizes. The Milnor $\varprojlim^1$ term vanishes. The resulting finite calculation in degree j uses only

$$j \leq v \leq m, \quad s = v - j \geq 0.$$

The only nonzero cohomology entries have $s = 0$, where the actual augmentation identifies them with the source homotopy modules. It therefore proves the truncated augmentation

$$\tau_{\leq m}((L_{K(n-1)}E_n)[1/p]) \xrightarrow{\sim} (\tau_{\leq m}(B[1/p]))^{h\mathcal{O}_{D_{n-1}}^\times}.$$

Taking the Postnikov limit of these already inverted spectra and using preservation of limits by fixed points gives

$$(L_{K(n-1)}E_n)[1/p] \xrightarrow{\sim} (B[1/p])^{h\mathcal{O}_{D_{n-1}}^\times}. \quad (\text{A.2})$$

It is the source unit, with its residual G_n -action.

The integral source descent [2, Proposition 4.6.13], equivalently Lemma 2.1, identifies $L_{K(n-1)}E_n$ with $B^{h\mathcal{O}_{D_{n-1}}^\times}$. Comparing that map with (A.2) proves that the canonical map $(B^{h\mathcal{O}_{D_{n-1}}^\times})[1/p] \rightarrow (B[1/p])^{h\mathcal{O}_{D_{n-1}}^\times}$ is an equivalence. The exact sequence

$$1 \longrightarrow \mathcal{O}_{D_{n-1}}^\times \longrightarrow G_{n,n-1} \longrightarrow G_n \longrightarrow 1$$

allows iterated fixed points. Combining these identifications with (A.1) proves the assertion for B .

Each comparison proved here is the canonical map obtained by moving the inversion telescope past the continuous bar diagram. The cotensor comparisons are natural in profinite sets, so they commute with pullback along a group homomorphism and with equivariant coefficient maps. This gives the two naturality squares for the actual maps into B . $\square$

B Adic descent for the deformation coefficients

Section 2 and Lemma A.6 use the following specialization of [2, Proposition 4.5.5 and Remark 4.5.6]. Its proof specifies the witnesses and augmentation maps for the three deformation problems.

Lemma B.1. *For E_n , E_{n-1} , and B , the pointwise deformation constructions satisfy the descent and cotensor assertions of [2, Proposition 4.5.5 and Remark 4.5.6]. Their comparison maps are natural in profinite parameters and in the specified maps of deformation problems.*

The deformation rings are flat and complete over their Noetherian local witnesses. The lifting from residue descent uses the following adic completion step in the proof of [2, Proposition 4.3.13] for the augmented module diagrams used below.

Lemma B.2. *Let A_0 be a Noetherian complete local ring with maximal ideal I_0 and residue field k_0 . Let $M^{-1} \rightarrow M^\bullet$ be an augmented cosimplicial diagram of flat, I_0 -adically complete A_0 -modules, with A_0 -linear structure maps. Suppose the residue augmentation is an equivalence in $D(k_0)$:*

$$M^{-1}/I_0 M^{-1} \xrightarrow{\sim} \mathrm{Tot}(M^\bullet/I_0 M^\bullet).$$

Then the augmentation $M^{-1} \rightarrow \mathrm{Tot} M^\bullet$ is an equivalence in $D(A_0)$.

Proof. Flatness gives, naturally for every cosimplicial degree a including $a = -1$,

$$I_0^r M^a / I_0^{r+1} M^a \simeq (M^a / I_0 M^a) \otimes_{k_0} (I_0^r / I_0^{r+1}).$$

The second factor is a finite-dimensional k_0 -vector space. Tensoring with it is a finite direct sum and hence commutes with totalization. The residue hypothesis therefore proves descent for every associated graded diagram.

For $r \geq 1$, the short exact sequences

$$0 \longrightarrow I_0^r M^a / I_0^{r+1} M^a \longrightarrow M^a / I_0^{r+1} M^a \longrightarrow M^a / I_0^r M^a \longrightarrow 0$$

are compatible with the augmentation and all cosimplicial maps. Totalization is exact in the stable derived category. Induction on r , starting with the residue hypothesis, proves

$$M^{-1}/I_0^r M^{-1} \xrightarrow{\sim} \mathrm{Tot}(M^\bullet/I_0^r M^\bullet).$$

The quotient towers have surjective transition maps. Their derived inverse limits equal their ordinary inverse limits, which are M^a by completeness. Finally, limits commute with limits, giving

$$\begin{aligned} M^{-1} &\simeq \varprojlim_r M^{-1}/I_0^r M^{-1} \\ &\simeq \varprojlim_r \mathrm{Tot}(M^\bullet/I_0^r M^\bullet) \\ &\simeq \mathrm{Tot} M^\bullet. \end{aligned}$$

□

Proof of Lemma B.1. Fix one of the three deformation problems and a profinite hypercover of its parameter set. Let $D^{-1} \rightarrow D^\bullet$ be the augmented diagram of degree-zero deformation rings. Let A_0 be the Noetherian witness of this chosen problem, with maximal ideal I_0 . For E_n and E_{n-1} , the pairs (A_0, I_0) are, respectively,

$$\begin{aligned} &(W(\overline{\mathbb{F}}_p)[[u_1, \dots, u_{n-1}]], (p, u_1, \dots, u_{n-1})), \\ &(W(\overline{\mathbb{F}}_p)[[u_1, \dots, u_{n-2}]], (p, u_1, \dots, u_{n-2})). \end{aligned}$$

Their residue field is $\overline{\mathbb{F}}_p$. For B , the witness is the ring in (2.7), with maximal ideal $(p, u_1, \dots, u_{n-2})$ and residue field $\overline{\mathbb{F}}_p((u_{n-1}))$. Deformation base change makes every term flat over A_0 and identifies its ideal of definition with $I_0 D^a$ [2, Proposition 4.2.4 and Lemma 4.2.12]. The rings are complete for that ideal [2, Remark 4.2.7 and Proposition 4.4.2]. The residue augmentation satisfies derived descent because the residue ring is the original solid coefficient sheaf [2, Proposition 4.5.1]. These are diagrams of A_0 -algebras, so all their maps are A_0 -linear. Localization and completion preserve Noetherianness. The completed ideal lies in the Jacobson radical, and its quotient is a field, so this witness is local. The identification of its ideal with the deformation kernel is [2, Definition 4.4.1 and Proposition 4.4.4]. The ensuing flat relatively perfect base changes preserve the ideal and all successive quotients. Lemma B.2 therefore proves descent for each degree-zero diagram.

The quotient step in the localized residue descent of [2, Lemma 4.4.5] uses a finite regular sequence. The displayed witnesses have regular parameters generating I_0 , so A_0/I_0 has a finite free Koszul resolution. Tensoring with this perfect module commutes with limits; hence

$$(A_0/I_0) \otimes_{A_0}^L \text{Tot } D^\bullet \simeq \text{Tot}((A_0/I_0) \otimes_{A_0}^L D^\bullet).$$

Termwise flatness, including that of the augmentation term, identifies these derived quotients with ordinary quotients. Before localization, the height ideal is generated by an initial segment of the source witness's regular parameters [2, Lemma 4.3.11], so the same argument applies to that reduction step.

After reduction, the cosimplicial terms are concentrated in degree zero and totalization is represented by their normalized cochain complex. Localization is exact and commutes with the finite kernels defining each normalized term. It therefore preserves the augmentation quasi-isomorphism, giving the required height- $n-1$ localized residue descent.

Let ω_0 be the invariant differential line of the witness deformation. The even homotopy modules in the diagram are $D^a \otimes_{A_0} \omega_0^{\otimes r}$ for $r \in \mathbb{Z}$, with compatible maps; odd homotopy modules vanish. The line is finite projective, so these modules are flat and I_0 -adically complete. Their residues satisfy descent by the residue calculation and tensoring with the one-dimensional residue line. Lemma B.2 proves descent for every homotopy module.

To pass to spectra, truncate every term above a fixed homotopy degree m . In output degree j , the totalization spectral sequence only uses

$$j \leq v \leq m, \quad s = v - j \geq 0.$$

This is a finite range. Equivalently, the fibers of the partial-totalization tower are bounded above by $m-s$, so the tower stabilizes in each fixed output degree. Homotopy-module descent thus proves descent for the truncated spectra. Postnikov completeness and commutation of inverse limits give descent for the original spectra.

The same adic filtration verifies solidity of the homotopy sheaves. Their residue sheaves are solid, each graded piece is a finite sum of residue modules, and the complete homotopy sheaves are limits of their finite filtered quotients. Solid modules are closed under these extensions and limits. All equivalences above are the original augmentations. Their naturality is unchanged, and applying the construction to products of profinite parameter sets gives the cotensor identifications of [2, Remark 4.5.6]. $\square$

C The actual completed frame field at every prime

The coefficient constructions in Section 2 use the completed frame field of Lemma C.1. Its proof computes the ramification of the connected-part frame field. The constant-field comparison of Lemma C.2 then supplies the base change used in Lemma D.11.

Throughout, p is any prime, $n \geq 3$. The field k is either $\overline{\mathbb{F}}_p$ or a finite field containing $\mathbb{F}_{p^{n(n-1)}}$. In particular both Honda endomorphism algebras used in the frame construction are defined over k .

Perfectness and the Laurent-perfection description below are the conclusions of [2, Proposition 2.7.1(1), (4)]. We give a Gross–Wintenberger proof, verifying the coefficient calculation over the finite constant fields used here.

Lemma C.1 (The completed frame field at every prime). *Put $K = k((u_{n-1}))$. Let $\mathcal{G}$ be the formal group obtained from the universal height- n deformation by setting $p, u_1, \dots, u_{n-2}$ equal to zero with u_{n-1} as the sole deformation parameter. Let L be the field generated over K by the coefficients of a connected-part frame, that is, an isomorphism over a separable closure between the generic formal group of $\mathcal{G}$ and the height- $n-1$ Honda formal group. Then the full frame torsor is connected and has coordinate field L . The extension L/K is infinite, totally ramified, Galois, and strictly arithmetically profinite, with Galois group $\mathcal{O}_{D_{n-1}}^\times$. Its completion $\widehat{L}$ is a perfectoid field of characteristic p , and there is a noncanonical isomorphism of topological k -fields*

$$\widehat{L} \simeq k(\widehat{(s)})^{\text{perf}}.$$

Every jointly continuous action on L by valuation-preserving field automorphisms extends uniquely to a jointly continuous action on $\widehat{L}$.

Proof. We recall the coefficient proof of [19, Theorem 3.5(1) and Lemmas 4.2–4.3]. Gross assumes algebraically closed constants; the calculation below also applies to the finite fields specified above. The special height is n , the generic height is $n-1$, and the leading coefficient of the p -series has valuation one. Choose compatible coordinates and write

$$\begin{aligned} [p]_{\mathcal{G}}(X) &= f(X^{p^{n-1}}), & f(T) &= \sum_{i \geq 1} a_i T^i, \\ a_1 &= u_{n-1}, & a_p &\equiv 1 \pmod{u_{n-1}}, & a_i &\in u_{n-1}k[[u_{n-1}]] \quad (i \neq p). \end{aligned}$$

These properties follow from the Honda special fibre $[p](X) = X^{p^n}$ and the deformation parameter u_{n-1} . Let $\phi(X) = \sum_{j \geq 1} \kappa_j X^j$ be an isomorphism from the generic formal group to the height- $n-1$ Honda group, whose p -series is $X^{p^{n-1}}$. Its coefficient field is unchanged if the inverse frame is used. The identity

$$\phi(f(X^{p^{n-1}})) = \phi(X)^{p^{n-1}} \tag{C.1}$$

shows successively that every κ_j is integral: the equation for κ_j is monic, with linear coefficient $-u_{n-1}^j$ and remaining coefficients polynomial in the earlier integral κ_i and the a_i .

Choose a uniformizer Π of $\mathcal{O}_{D_{n-1}}$. For $m \geq 0$, define K_m to be the fixed field of the kernel of the frame representation modulo $1 + \Pi^{m+1}\mathcal{O}_{D_{n-1}}$, and put $K_{-1} = K$. These fields are finite Galois over K , without any surjectivity assertion yet. The coefficient description in [19, Lemma 4.2] gives

$$K_0 = K(\kappa_1), \quad \kappa_j \in K_{m-1} \ (j < p^m), \quad K_m = K_{m-1}(\kappa_{p^m}) \quad (m \geq 1). \tag{C.2}$$

Here only the Honda automorphism filtration is used: an automorphism in $1 + \Pi^m\mathcal{O}_{D_{n-1}}$ is detected modulo $1 + \Pi^{m+1}\mathcal{O}_{D_{n-1}}$ by its coefficient in degree p^m . All these automorphisms are defined over k , because $\mathbb{F}_{p^{n-1}} \subset k$. Thus this description, and the integrality argument just given, hold over the stated finite fields as well as over algebraically closed constants.

The first coefficient equation is $\kappa_1^{p^{n-1}-1} = u_{n-1}$. It is Eisenstein and separable, and $\mathbb{F}_{p^{n-1}}^\times \subset k$ supplies its roots of unity. Consequently K_0/K is totally tamely ramified of

degree $p^{n-1} - 1$, with uniformizer κ_1 . Suppose inductively that K_{m-1}/K is totally ramified of degree $(p^{n-1} - 1)p^{(n-1)(m-1)}$ and that $\kappa_{p^{m-1}}$ is a uniformizer. The coefficient of $X^{p^{n-1}+m}$ in (C.1) gives

$$y^{p^{n-1}} - u_{n-1}^{p^m} y - b_m = 0, \quad y = \kappa_{p^m}, \quad b_m = \sum_{j=1}^{p^m-1} \kappa_j [T^{p^m}] f(T)^j. \quad (\text{C.3})$$

All terms of b_m lie in K_{m-1} by (C.2). The only monomial in this sum that uses exclusively a_p is $\kappa_{p^{m-1}} a_p^{p^{m-1}}$, and it has K_{m-1} -valuation one. Every other monomial contains an a_i with $i \neq p$, so has valuation at least

$$v_{K_{m-1}}(u_{n-1}) = (p^{n-1} - 1)p^{(n-1)(m-1)} > 1.$$

The last strict inequality holds because $n \geq 3$, including $p = 2$. Thus $v_{K_{m-1}}(b_m) = 1$ and (C.3) is a separable Eisenstein equation of degree p^{n-1} . It follows that K_m/K_{m-1} is totally ramified of degree p^{n-1} , with uniformizer y . The polynomial splits already in K_m : its roots are

$$y + c\kappa_1^{p^m}, \quad c \in \mathbb{F}_{p^{n-1}}.$$

This also identifies its Galois group with the additive group of $\mathbb{F}_{p^{n-1}}$ and makes the finite-constant-field assertion explicit.

Induction proves

$$[K_m : K] = (p^{n-1} - 1)p^{(n-1)m}, \quad \text{Gal}(K_m/K) = \mathcal{O}_{D_{n-1}}^\times / (1 + \Pi^{m+1} \mathcal{O}_{D_{n-1}}). \quad (\text{C.4})$$

The second assertion follows by comparing the degree with the order of the group into which the frame representation injects. Its image is therefore full at every finite level, and hence full in $\mathcal{O}_{D_{n-1}}^\times$. The coefficient description also gives $L = \bigcup_m K_m$. Thus this is the actual full frame field, and not a component obtained after a constant extension. All its finite stages have residue field exactly k .

For any nonidentity $\sigma \in \text{Gal}(K_m/K_{m-1})$, the displayed roots give

$$v_{K_m}(\sigma(y) - y) = p^m v_{K_m}(\kappa_1) = p^{nm}.$$

Accordingly K_m/K_{m-1} has the single lower, and hence upper, ramification break

$$i_m = p^{nm} - 1 \quad (m \geq 1). \quad (\text{C.5})$$

This recovers Gross's break calculation directly over k . Herbrand's formula, or the elementary-tower criterion [20, §1.4.2], gives the positive upper breaks of L/K :

$$a_m = \frac{p^n - 1}{(p^{n-1} - 1)(p - 1)} (p^m - 1).$$

Indeed $a_1 = i_1/(p^{n-1} - 1)$ and $a_m - a_{m-1} = (i_m - i_{m-1})/[K_{m-1} : K]$ for $m \geq 2$. These breaks tend to infinity, so L/K is APF. Moreover,

$$\frac{i_m}{[K_m : K]} = \frac{p^m - p^{-(n-1)m}}{p^{n-1} - 1} \geq \frac{p - p^{-(n-1)}}{p^{n-1} - 1} > 0. \quad (\text{C.6})$$

By [20, §1.4.1], this is strict APF. The denominator in that criterion is the degree of the upper field of the elementary step; its initial unramified field is our K , its initial tame field is our K_0 , and the step with break i_m ends at our K_m . In particular the totally ramified p -extension L/K_0 is also strictly APF.

We next justify perfectness without assuming it in the field-of-norms construction. Wintenberger's local fields may have any perfect residue field of characteristic p . Normalize the valuation of K_0 to have value group $\mathbb{Z}$, and let $X = X_{K_0}(L)$ be the field of norms. By [20, Theorem 2.1.3 and Proposition 4.2.1], X is a complete discretely valued field of characteristic p , with residue field k , and there is a valuation-preserving embedding

$$X \longrightarrow R(\widehat{L}) := \varprojlim_{x \mapsto x^p} \widehat{L}.$$

In characteristic p the operations in this inverse limit are coordinatewise. Frobenius is invertible, with inverse the left shift, and projection to coordinate zero is injective. The valuation is $v_R(x) = v(x_0)$. Since $v(x_r - y_r) = p^{-r}v(x_0 - y_0)$, a Cauchy sequence has compatible limits in every coordinate. Hence $R(\widehat{L})$ is complete and perfect, and its isometric embedding into $\widehat{L}$ has closed image. This is the equal-characteristic case of [20, Theorem 4.1.2 and Remark 4.1.3.3]; it does not presuppose perfectness of $\widehat{L}$. The embedding of X therefore extends to an isometric embedding

$$\widehat{X^{\text{perf}}} \longrightarrow \widehat{L}.$$

The precise approximation input is [20, §4.3.3.1], with $M = L$: for every $a \in \mathcal{O}_{\widehat{L}}$ there is $\delta \in \widehat{\mathcal{O}_{X^{\text{perf}}}}$ with

$$v(a - \delta) \geq \gamma, \quad \gamma = \frac{p-1}{p}c(L/K_0) > 0. \quad (\text{C.7})$$

Here c is the strict APF constant. The proof uses the finite-level approximation in [20, Remark 2.3.3.1] and the norm-versus-power estimate in [20, Lemma 4.2.2.1]. Its general bound is $\min\{v(p), (p-1)c(L/K_0)/p\}$, which is the displayed positive number in characteristic p . Choose $\beta \in \widehat{X^{\text{perf}}}$ with $0 < v(\beta) \leq \gamma$, using a p -power root of a uniformizer of X . Starting with any $a_0 \in \mathcal{O}_{\widehat{L}}$, apply (C.7) repeatedly and set

$$a_{j+1} = \beta^{-1}(a_j - \delta_j), \quad \delta_j \in \widehat{\mathcal{O}_{X^{\text{perf}}}}.$$

All a_j remain integral, and

$$a_0 = \sum_{j=0}^{N-1} \beta^j \delta_j + \beta^N a_N, \quad v(\beta^N a_N) \geq Nv(\beta) \longrightarrow +\infty.$$

Completeness of $\widehat{X^{\text{perf}}}$ proves that a_0 lies in its image. Thus $\widehat{X^{\text{perf}}} = \widehat{L}$; this is also the equal-characteristic conclusion of [20, Corollary 4.3.4]. The valuation is nondiscrete by the unbounded degrees in (C.4). A complete nondiscretely valued perfect field of characteristic p is perfectoid.

The preceding identification respects the original constants, not merely their residue classes. For $c \in k$ and a finite field $K_0 \subset E \subset L$, put $e_E = [E : K_0]$, a power of p . The coefficient embedding of [20, §2.1.2] is represented by the norm-compatible constants

$$c_E = c^{1/e_E} \in k \subset E.$$

In the embedding of [20, Proposition 4.2.1], its zeroth coordinate is the limit of $c_E^{e_E} = c$. Thus the embedding $\widehat{X^{\text{perf}}} \rightarrow \widehat{L}$ is a k -embedding. Choosing a uniformizer s of X now gives $X \simeq k((s))$ and the claimed isomorphism of topological k -fields after perfection and completion. No equivariance is asserted for this choice of s .

Finally, a valuation-preserving field automorphism is an isometry, and so extends uniquely to the completion. The extensions preserve composition. Joint continuity follows by approximating each element of $\widehat{L}$ by an element of L : all group elements preserve the error of approximation, while the orbit map of the approximating element is continuous. $\square$

For the actions furnished by the frame construction, the source stabilizer automorphisms preserve the maximal ideal of the one-parameter deformation ring, and hence send u_{n-1} to a unit multiple of u_{n-1} ; the target automorphisms act over K ; and arithmetic descent preserves the u_{n-1} -adic valuation. The unique extension of this valuation to each finite coefficient field is consequently preserved by these actions. Their continuous action on the frame tower therefore extends to the completion as in the lemma.

Lemma C.2 (Actual extension of the constant field). *Let k_0 be finite and contain $\mathbb{F}_{p^{n(n-1)}}$, and let k' be a finite extension of k_0 or $\overline{\mathbb{F}}_p$. Form the frame fields using compatible frames for the deformation over $k_0[[u_{n-1}]]$ and its extension to $k'[[u_{n-1}]]$. The natural map induces an isomorphism*

$$\widehat{L}_{k_0} \widehat{\otimes}_{k_0} k' \xrightarrow{\sim} \widehat{L}_{k'}.$$

The tensor product is completed for the topology induced by the open k_0 -subspaces of $\widehat{L}_{k_0}$. This is the map of the actual frame construction and is compatible with its source, target, and arithmetic actions; it does not use a choice of Laurent parameter.

Proof. Write $K_m(k)$ for the finite frame fields in the proof above. Their coefficient equations are preserved by extension of constants, and remain separable Eisenstein equations of the same degrees. In particular, if $\pi_m = \kappa_{p^m}$, then

$$K_m(k_0) = k_0((\pi_m)), \quad K_m(k') = k'((\pi_m)), \quad K_m(k_0) \widehat{\otimes}_{k_0} k' = K_m(k').$$

These identifications use the chosen compatible frame only to verify the natural map, and are compatible as m increases.

The residue field of $\widehat{L}_{k_0}$ is k_0 , since completion does not change the residue field. For any finite intermediate field $k_0 \subset k_1 \subset k'$, choose a k_0 -basis $e_1, \dots, e_d$ of k_1 . For $b_i \in \widehat{L}_{k_0}$, residue-field linear independence gives

$$v\left(\sum_i b_i e_i\right) = \min_i v(b_i).$$

Indeed, after dividing by a nonzero b_i of minimum valuation, the reduction is a nonzero k_0 -linear combination of the residues of the e_i . Thus the natural map $\widehat{L}_{k_0} \otimes_{k_0} k_1 \rightarrow \widehat{L}_{k'}$ is injective and gives precisely the tensor valuation topology. Passing through all finite k_1 proves the same assertion for $\widehat{L}_{k_0} \otimes_{k_0} k'$. Its image is dense in every $k'((\pi_m))$: finite Laurent polynomials use only finitely many constants of k' and approximate every Laurent series. The union of these fields is $L_{k'}$, which is dense in $\widehat{L}_{k'}$. The natural map consequently becomes an isomorphism after completion.

The frame equations and scalar-extension maps are natural in the deformation and in its frame. Therefore the maps just used commute with the source and target actions and with arithmetic descent. Equivariance extends from this dense subalgebra to the completions by continuity. In particular, arithmetic Frobenius acts on the scalar $c \in k'$ by its original action on c ; no Frobenius twist is introduced by the field-of-norms identification. $\square$

D Integral frames and the coheight-one comparison

The test-family construction in Lemma 4.3 uses the integral frame and determinant comparisons proved here, with their common arithmetic action. Lemma 2.7 uses the norm-one quotient and its residue cohomology over the required constant fields.

The section-lifting and triangular-matrix arguments supply proof steps for [16, Proposition 5.16]; the completed scalar-extension argument supports the residue-cohomology

comparison. Throughout this appendix, p is any prime, $n \geq 3$, and k is either $\overline{\mathbb{F}}_p$ or a finite field containing $\mathbb{F}_{p^{n(n-1)}}$. Put

$$X_0 = \Gamma_{n-1} \oplus \mathbb{Q}_p/\mathbb{Z}_p, \quad N_0 = M(X_0).$$

Here M is the usual covariant integral Dieudonné functor, with operators $F : \varphi^*M \rightarrow M$ and $V : M \rightarrow \varphi^*M$ satisfying $FV = VF = p$. The Frobenius used for shtukas and Fargues–Fontaine bundles is instead

$$\Phi = p^{-1}F, \quad p\Phi = F, \quad \Phi^{-1} = V. \quad (\text{D.1})$$

Thus $F = p\varphi$ on the covariant module of $\mathbb{Q}_p/\mathbb{Z}_p$, whereas $F = \varphi$ on that of μ_{p^∞} ; their normalized Frobenii are respectively φ and $p^{-1}\varphi$. This is the normalization of [15, p. 99, footnote 1, and Theorem 17.5.2]. With it, the height- i , dimension-one Honda group gives the positive bundle $\mathcal{O}(1/i)$, in the convention of [16, Definition 5.1]. Multiplication of Frobenius by p^{-1} changes no endomorphism or model-frame action. All generic framings below intertwine these normalized Frobenii; for integral framings this is equivalent to intertwining F and V . The Honda groups and their arithmetic descent data are those used in the main text. References to [16] use its second arXiv version. Write

$$G_{n,n-1}^0(k) = (D_n^\times \times D_{n-1}^\times) \rtimes \text{Gal}(k/\mathbb{F}_p),$$

with the common arithmetic component. For $k = \overline{\mathbb{F}}_p$ this is $G_{n,n-1}^0$ of the main text. The constant-field hypothesis ensures that all Honda endomorphisms used here are defined over k . Let $\widehat{L}_k$ denote the completion of the corresponding Gross–Torii connected-part frame field over $k((u_{n-1}))$. Lemma C.1 proves that it is perfectoid and gives, compatibly with its constant field,

$$\widehat{L}_k \simeq \bigcup_{r \geq 0} \widehat{k((s^{1/p^r}))}.$$

D.1 Integral and meromorphic framed objects

Let $S = \text{Spa}(R, R^+)$ be affinoid perfectoid over k . The notation $Y_{S, \langle 0, \infty \rangle}$ denotes the gluing category of [2, Definition 2.5.2]. For a shtuka or meromorphic bundle $\mathcal{H}$, write $\text{gen}(\mathcal{H})$ for its rational completion at $p = 0$ and $\text{sp}(\mathcal{H})$ for its associated vector bundle on X_S . Write $\mathcal{E}_i = \mathcal{O}_{X_S}(1/i)$ for the bundle determined by the chosen height- i Honda model, for $i = n, n-1$.

Let $\mathcal{Z}^{\text{int}}$ be the v -sheafification of the groupoid of data $(\mathcal{E}, \Phi, b, a)$, where $(\mathcal{E}, \Phi)$ is a shtuka on the integral space $Y_{S, [0, \infty)}$, with paw at $p = 0$, whose lattice is preserved by $p\Phi$ and Φ^{-1} ,

$$b : N_0 \otimes_{W(k)} W(R) \xrightarrow{\sim} \widehat{\mathcal{E}}_p$$

is an *integral* isomorphism intertwining $p^{-1}F$ with Φ , and $a : \mathcal{E}_n \xrightarrow{\sim} \text{sp}(\mathcal{E})$ is a special frame. Let $\mathcal{Z}^{\text{mer}}$ instead classify $(\mathcal{V}, \Phi, b, a)$ with $\mathcal{V}$ meromorphic and

$$b : N_0 \otimes_{W(k)} W(R)[1/p] \xrightarrow{\sim} \text{gen}(\mathcal{V}), \quad a : \mathcal{E}_n \xrightarrow{\sim} \text{sp}(\mathcal{V}).$$

No additional integral lattice is retained in this second definition.

The following framed form of [2, Lemma 2.5.4(1)] keeps track of the chosen integral generic lattice.

Lemma D.1. *Restriction gives a natural equivalence $\mathcal{Z}^{\text{int}} \simeq \mathcal{Z}^{\text{mer}}$. It commutes with base change, with the compact integral frame groups, and with the common arithmetic descent datum.*

Proof. The Frobenius-compatible Beauville–Laszlo equivalence is

$$\mathrm{VB}(Y_{S,[0,\infty)}) \simeq \mathrm{VB}(Y_{S,\langle 0,\infty \rangle}) \times_{\mathrm{Proj}(W(R)[1/p])} \mathrm{Proj}(W(R));$$

see [2, Lemma 2.5.4(1)]. Glue a meromorphic object to $N_0 \otimes W(R)$ using b . The operators $p\Phi$ and Φ^{-1} glue to F and V on this lattice, respectively. Their composites are multiplication by p . This gives the required integral lattice for the normalized shtuka, with its specified integral frame. Restriction recovers the original meromorphic object. The equivalence on morphisms proves that the two constructions are inverse and commute with base change.

An integral frame change preserves N_0 . A rational frame change acts on $\mathcal{Z}^{\mathrm{mer}}$ and acts on $\mathcal{Z}^{\mathrm{int}}$ by this same construction with the new gluing map. In particular, a rational frame change need not preserve a previously chosen integral lattice. The Honda arithmetic descent maps preserve the integral modules, so the same naturality proves compatibility with their semilinear action. $\square$

Here the Rapoport–Zink problem means a dimension-one, height- n p -divisible group X over R^+ , together with a special quasi-isogeny from Γ_n over R^+/ϖ ; the functor is independent of the chosen pseudo-uniformizer after passage to the v -site. Its coheight-one stratum requires connected height $n - 1$ and étale height one on the generic fiber. The framed problem also remembers an integral isomorphism $(X_0)_R \simeq X_R$. We allow all heights of the special quasi-isogeny until the height-zero component is specified. We write $\mathrm{LT}_{n,n-1}^\infty$ for the height-zero problem with both these integral generic frames. Thus “integral” refers to the generic Dieudonné lattice as well as to the original p -divisible group.

This is the normalized, integrally framed form of [2, Proposition 2.4.2]. We spell out the lattice condition and its compatibility with the model frames.

Lemma D.2. *With (D.1), the integral Rapoport–Zink moduli problem and its generic-frame moduli problem identify naturally with the corresponding normalized shtuka problems. These identifications commute with base change, model automorphisms and the common arithmetic descent datum. On the coheight-one stratum the generic integral module has type N_0 at every geometric point of $\mathrm{Spec} R$.*

Proof. We specify both directions of the comparison, including its integral condition. For a p -divisible group X over R^+ , restrict $M(X)$ from $\mathrm{Spec} W(R^+)$ to $Y_{S,[0,\infty)}$ and use $\Phi = p^{-1}F$. Its generic integral module is $M(X_R)$. Its special quasi-isogeny from Γ_n gives the frame $a : \mathcal{E}_n \simeq \mathrm{sp}(\mathcal{E})$ by crystalline base change and restriction. The normalized rational covariant Honda module has slope $-1/n$ in the usual isocrystal convention and hence gives $\mathcal{E}_n = \mathcal{O}(1/n)$ in the bundle convention being used.

For the converse it suffices to work on products of points, a basis for the v -topology, and first replace R^+ by R° . Use the characteristic- p untilt S itself, so the primitive element defining the paw is $\xi = p$. The local-shtuka/Breuil–Kisin–Fargues comparison [16, Corollary 1.9] then extends $(\mathcal{E}, \Phi)$ to a finite projective $W(R^\circ)$ -module M with rational Frobenius. Its integral completion at p is the given generic Dieudonné module. Beauville–Laszlo gluing along p therefore implies that $p\Phi$ and Φ^{-1} preserve $M[1/[\varpi]]$. They already preserve $M[1/p]$. The pair $(p, [\varpi])$ is a regular sequence in $W(R^\circ)$: modulo p , the second element is the non-zero-divisor ϖ in R° . Consequently, after any affine localization on which M is free,

$$W(R^\circ)[1/[\varpi]] \cap W(R^\circ)[1/p] = W(R^\circ)$$

inside $W(R^\circ)[1/(p[\varpi])]$, and the same identity holds after that localization. Thus $F = p\Phi$ and $V = \Phi^{-1}$ preserve M and satisfy $FV = VF = p$. Integral Dieudonné theory over the perfect ring R° produces X ; equivalently one may apply the normalized classification in [15,

Theorem 17.5.2]. In the framed coheight-one problem its generic module is N_0 , so its generic dimension is one. Dimension of a p -divisible group is locally constant, and $\mathrm{Spec} R \subset \mathrm{Spec} R^\circ$ is dense because ϖ is a non-zero-divisor. Hence X has dimension one everywhere.

Both the constructed object and the constant Honda object extend to $Y_{S,(0,\infty]}$. The full faithfulness of restriction [16, Proposition 7.3(1)] lifts the frame a , together with its inverse, to this larger space. The equivalence with Frobenius modules over $B_{\mathrm{cris}}^+(R^\circ/\varpi)$ is the ring-restriction construction of [2, Lemma 2.3.3(1)]; replacing F throughout by $p^{-1}F$ changes neither its morphisms nor this argument. Full faithfulness of Dieudonné theory up to isogeny over the f -semiperfect ring R°/ϖ [2, Theorem 2.3.2(1)] then gives the required special quasi-isogeny. This is the Rapoport–Zink object. Here R°/ϖ is f -semiperfect because it is the quotient of a perfect ring by a principal ideal. We use the crystalline full-faithfulness assertion after inverting p ; no assertion that A_{cris} is p -torsion free is needed. The formal Lubin–Tate description identifies the resulting functors on R° and R^+ ; on the characteristic- p coheight-one chart it simply asks that the deformation parameter be topologically nilpotent. The same holds in every height component by source quasi-isogenies.

These two constructions are inverse by their fully faithful linear-algebra descriptions. To check the asserted integral type on the coheight-one chart, use the universal coordinates: $u_1 = \cdots = u_{n-2} = 0$ and u_{n-1} is invertible in R . These are algebraic equations and an algebraic invertibility condition. They therefore hold at every geometric point of $\mathrm{Spec} R$, including points not arising from a continuous valuation. Every such fiber has connected height $n-1$ and étale height one. Over its algebraically closed residue field it is integrally isomorphic to $\Gamma_{n-1} \oplus \mathbb{Q}_p/\mathbb{Z}_p$. Conversely these fiber conditions give the same height stratum; on a perfect ring, vanishing at all geometric points is vanishing, and nonvanishing at all maximal ideals is invertibility. All constructions used are natural restriction, crystalline base change, gluing, or unique fully faithful lifts. This proves the stated equivariance, including its coherent compatibilities for parameterized continuous actions. $\square$

D.2 The compact integral frame torsor

This is the compact integral-frame assertion of [2, Lemma 2.4.3]. We verify its local triviality using integral Dieudonné theory over valuation rings.

Lemma D.3. *Let N be an integral Dieudonné module over the perfect ring R . Suppose that at every geometric point of $\mathrm{Spec} R$ it is integrally isomorphic to N_0 . Then the sheaf*

$$\mathrm{Isom}_{F,V}(N_0 \otimes W(R), N)$$

on perfectoid base changes is a pro-finite étale right $(\mathcal{O}_{D_{n-1}}^\times \times \mathbb{Z}_p^\times)$ -torsor. For the integral Rapoport–Zink shtuka moduli problem, these are the fibers of the forgetful map from $\mathcal{Z}^{\mathrm{int}}$. Its restriction to the height-zero component is the actual infinite-level Lubin–Tate frame tower.

Proof. Integral Dieudonné theory over perfect rings identifies N with a p -divisible group X ; we use [2, Theorem 2.3.2(2)]. We verify local existence of frames on the product-of-points basis of the v -topology. On such a base write

$$R = \left(\prod_{i \in I} C_i^+ \right) [1/\varpi],$$

where C_i^+ are valuation rings with algebraically closed fraction fields C_i . Every local ring of R is a valuation domain with algebraically closed fraction field. Indeed, a prime of the product ring determines an ultrafilter by its choices of idempotents. Its local ring is a

localization of the corresponding ultraproduct valuation ring, whose fraction field is $\prod_{\mathcal{U}} C_i$. This field is algebraically closed: a root of a polynomial of fixed degree can be chosen componentwise. Localization at ϖ preserves this description; compare [16, Proposition 2.18].

For each algebraic stalk $V = R_{\mathfrak{p}}$, put $K_V = \text{Frac}(V)$. The hypothesis gives an integral isomorphism $X_{K_V} \simeq (X_0)_{K_V}$. Both groups have constant Newton polygon on V . The valuation-ring full-faithfulness theorem [18, Lemma 4.2.16] extends the isomorphism and its inverse to V . Injectivity of restriction makes their compositions the identities. Thus $X_V \simeq (X_0)_V$. Only the valuation-ring statement is used here.

On perfect test schemes the automorphism sheaf of X_0 is the constant profinite group

$$\text{Aut}(X_0) = \mathcal{O}_{D_{n-1}}^{\times} \times \mathbb{Z}_p^{\times}.$$

The two distinct slope blocks have no rational homomorphisms, and hence no integral cross terms. On each block, solving the full Honda Frobenius equations gives the indicated integral endomorphism ring. Its reductions give locally constant finite-level functions, and their compatible inverse limit gives the profinite group sheaf. These are the reductions of integral endomorphisms, rather than arbitrary automorphisms of a truncated group. This is the constant-model computation in [11, Proposition 2.2.6].

Let $T = \text{Isom}(X_0, X)$ on perfect R -schemes. It is affine: take the inverse limit of the finite-level isomorphism schemes and then restrict to perfect schemes. Composition gives the functorial identity $T \times (\mathcal{O}_{D_{n-1}}^{\times} \times \mathbb{Z}_p^{\times}) \simeq T \times_{\text{Spec } R} T$. At every local ring of R , the preceding isomorphism trivializes T . Thus its affine structural map is faithfully flat. For every open normal subgroup $\mathcal{K} \subset \mathcal{O}_{D_{n-1}}^{\times} \times \mathbb{Z}_p^{\times}$, faithfully flat descent constructs a finite étale quotient $T/\mathcal{K}$, locally the finite constant scheme $(\mathcal{O}_{D_{n-1}}^{\times} \times \mathbb{Z}_p^{\times})/\mathcal{K}$, and

$$T \simeq \varprojlim_{\mathcal{K}} T/\mathcal{K}.$$

Passing from finite étale R -algebras to finite étale perfectoid covers and taking the inverse limit gives the asserted pro-finite étale torsor. The construction and torsor identity commute with base change, so they descend from this v -basis. Neither this finite-quotient descent nor the valuation-ring full-faithfulness argument requires this automorphism group to be free of p -torsion.

Finally Lemma D.2 identifies these frames with isomorphisms of the generic p -divisible group to its fixed model. Over the height-zero component this is exactly the connected-part and étale-part frame tower defining $\text{LT}_{n,n-1}^{\infty}$. $\square$

D.3 Lifting sections for filtered full faithfulness

We record the analytic lifting argument with its coefficient rings. For a perfect uniform Banach $\mathbb{F}_p$ -algebra R with power-multiplicative norm α , use the rings of [17, Hypothesis 5.0.1 and Definition 5.1.1]:

$$B_r = \left\{ \sum_{i \geq 0} p^i [a_i] \in W(R) : p^{-i} \alpha(a_i)^r \rightarrow 0 \right\}.$$

The analytic ring on the punctured radius interval is the Fréchet completion of $B_r[1/p]$ for the norms

$$\|z\|_u = \sup_i p^{-i} \alpha(a_i)^u, \quad 0 < u \leq r,$$

initially defined on the bounded subring. These norms extend to the completion, are sub-multiplicative, and satisfy $\|\varphi(z)\|_u = \|z\|_{pu}$. Choose the norm so that a pseudo-uniformizer

ϖ satisfies $\alpha(\varpi) = p^{-1}$ and $\alpha(\varpi^{-1}) = p$. Such a power-multiplicative norm defining the same topology exists by [17, Remark 2.8.18], followed by rescaling. Submultiplicativity in both directions then gives $\alpha(a\varpi^m) = p^{-m}\alpha(a)$ for every $m \in \mathbb{Z}$. For $r = p^N$ with $N \in \mathbb{Z}$, perfectness gives the same equality for the p -power roots of ϖ , so the coefficient condition above is exactly that of the endpoint ring $B_{S,[0,r]}$ in [2, (2.5.1)]. As $N \rightarrow -\infty$, these radii are cofinal among windows shrinking towards zero, and we use them for the geometric windows below. Characteristic- p perfectoid affinoids, including the product-of-points bases, satisfy the Banach-ring hypotheses with this choice of norm. Moreover, B_r is complete for $\|-\|_r$ by [17, Remark 5.1.8].

Full faithfulness in [16, Proposition 5.16] requires comparison of actual sections. For a nonpositive-slope bundle $\mathcal{H}$, vanishing of $H^0(\mathcal{H}_{<0})$ gives an injection into $H^0(\mathrm{gr}_0 \mathcal{H})$; the boundary to $H^1(\mathcal{H}_{<0})$ still controls which sections lift. The reduction to the slope-zero quotient in the cited proof does not settle this lifting condition. The following boundedness argument supplies the required comparison.

Lemma D.4. *Let $\mathcal{H}$ be a meromorphic bundle over an affinoid perfectoid characteristic- p base. Suppose it has a finite increasing filtration by subbundles whose graded pieces have pure generic slopes at most zero. Then restriction induces an isomorphism of sheaves on the base v -site*

$$\mathrm{Hom}_{\mathrm{mer}}(\mathcal{O}, \mathcal{H}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{FF}}(\mathcal{O}, \mathrm{sp}(\mathcal{H})).$$

Proof. Restriction is injective. Work v -locally and partition into clopen pieces on which the finitely many graded ranks are fixed. If every graded slope is strictly negative, both sides vanish by the pure-slope comparison and induction along the filtration. Otherwise $\mathcal{H}^\vee$ has constant generic Newton polygon with minimum slope zero. Proposition 5.9 of [16] provides an anti-effective shtuka lattice for $\mathcal{H}^\vee$. Its dual is an effective lattice for $\mathcal{H}$. This input uses the earlier gluing square and generic isocrystal standardization in that reference.

For an integral matrix D over B_{r_0} , put $C = \max(1, \|D\|_{r_0})$. Entrywise, for $0 < u \leq r_0$,

$$p^{-i}\alpha(a_i)^u = (p^{-i}\alpha(a_i)^{r_0})^{u/r_0} p^{-i(1-u/r_0)} \leq C^{u/r_0} \quad (i \geq 0).$$

Hence $\|D\|_u \leq C^{u/r_0}$. This estimate also gives a basis of the effective lattice near $p = 0$. Represent it on an endpoint window by a finite projective B_{r_0} -module M_{r_0} . After a base cover, M_{r_0}/pM_{r_0} is free over $B_{r_0}/p = R$. Lift a basis and its dual inverse to maps $\beta : B_{r_0}^d \rightarrow M_{r_0}$ and $\beta' : M_{r_0} \rightarrow B_{r_0}^d$. Both composition errors are divisible by p . Use a finite-projective summand presentation to regard these as matrices. On a sufficiently small endpoint window their norms are at most $p^{-1}C^{u/r_0} < 1$; Neumann series make both compositions invertible. Thus β is an actual basis there.

Choose a window strictly inside this basis window so that its Frobenius pullback is also defined. In the resulting coordinates the Frobenius matrix A has meromorphic entries and its completion has entries in $W(R)$. Therefore, after shrinking once more, $A \in M_d(B_r)$: the intersection of $B_r[1/p]$ with $W(R)$ is B_r , as follows from the bounded Witt expansions. Put $C = \max(1, \|A\|_r)$.

An actual section of $\mathrm{sp}(\mathcal{H})$ gives an analytic vector x on the punctured window with $x = A\varphi(x)$. The endpoint bounds, or [17, Lemma 5.2.1], give

$$M = \sup_{r/p \leq u \leq r} \|x\|_u < \infty.$$

For $0 < u < r/p$, choose m so that $r/p \leq p^m u \leq r$. Iteration gives

$$\|x\|_u \leq C^{(u+pu+\dots+p^{m-1}u)/r} \|x\|_{p^m u} \leq C^{1/(p-1)} M.$$

All radii used lie inside the chosen Frobenius window. The bounded-subring criterion [17, Lemma 5.2.2] now places every entry of x in $\bigcup_{s>0} B_s[1/p]$. A common smaller s works for the finitely many entries. Thus x is meromorphic at $p = 0$; the same Frobenius equation extends it along all the remaining translates. Uniqueness permits descent from the basis and the base cover.

The criterion is applied to the analytic element itself. Only the already bounded entries of A were expanded in Witt coordinates. In particular, no Witt expansion of an arbitrary analytic element is assumed; see [17, Remark 5.2.7]. $\square$

The same Hom comparison holds after a finite unramified coefficient-field extension of degree s . After extending the base constants to contain $\mathbb{F}_{p^s}$, use the same Witt coefficient rings and replace φ by φ^s . For $q = p^s$, the norm identity is then $\|\varphi^s(z)\|_u = \|z\|_{qu}$ and the iteration bound is $C^{1/(q-1)}M$. The bounded-subring criterion is unchanged. Thus the full-faithfulness assertion used to descend objects from such coefficient-field extensions follows from this same argument.

D.4 Triangular Frobenius matrices and filtered comparison

Lemma D.5 ([16, Lemma 5.20]). *Use the annular notation of [16, §5], allowing a finite unramified coefficient-field extension, with uniformizer π and its corresponding Frobenius φ . Let A be upper triangular over $B_{[1,1]}$, with diagonal $\pi^{s_1}, \dots, \pi^{s_d}$ and $s_1 \geq \dots \geq s_d$. There is a unipotent upper triangular matrix U over $B_{[1,\infty]}$ such that $U^{-1}A\varphi(U)$ has entries in $B_{[0,1]}[1/\pi]$.*

Proof. We include the argument with the scalar factor omitted in the last elimination equation of the cited proof made explicit. Induct first on the lower-right block and then from left to right in the first row. At the $(1, j)$ step, write

$$a_{1j} = a_{1j}^{\text{mer}} + a'_{1j}, \quad a'_{1j} \in [\varpi]B_{[1,\infty]},$$

using [16, Lemma 5.17]. For $U = 1 + yE_{1j}$, direct multiplication gives

$$(U^{-1}A\varphi(U))_{1j} = a_{1j} + \pi^{s_1}\varphi(y) - \pi^{s_j}y.$$

Choose y satisfying

$$\pi^{-(s_1-s_j)}y - \varphi(y) = \pi^{-s_1}a'_{1j}.$$

The transformed entry is then a_{1j}^{mer} . It exists by [16, Lemma 5.19], since $s_1 - s_j \geq 0$ and $[\varpi]B_{[1,\infty]}$ is stable under powers of π . The lower-right block and the earlier first-row entries stay fixed. The induction therefore terminates with the desired matrix. $\square$

An increasing slope filtration is a finite filtration by subbundles, indexed by rational numbers $\lambda_1 < \dots < \lambda_r$, whose λ_j -graded piece is pure of slope λ_j . For a meromorphic bundle, purity here refers to its generic isocrystal, with the slope convention of [16, Definition 5.1].

The first assertion below is [16, Proposition 5.16]; the framed assertion is its consequence using [16, Propositions 5.13 and 5.22]. We give the proof with the section-lifting and triangular elimination steps supplied above.

Lemma D.6. *The special-fiber functor gives an equivalence between the v -stacks of meromorphic bundles with increasing slope filtrations and vector bundles on X_S with such filtrations. Moreover, the meromorphic objects with generic isocrystal framed by $(N_0[1/p], p^{-1}F)$ correspond to filtered bundles with graded pieces framed by $\mathcal{O} \oplus \mathcal{E}_{n-1}$.*

Proof. For two filtered objects, the nonpositive part of their internally filtered Hom bundle represents filtration-preserving morphisms. Indeed, a source piece of slope λ can map only to target pieces of slope $\mu \leq \lambda$ in this part, whose graded Hom pieces have slopes $\mu - \lambda \leq 0$. Lemma D.4 therefore proves full faithfulness by lifting actual morphisms after applying sp.

For essential surjectivity, make an unramified coefficient-field extension clearing the finitely many slope denominators. The descent step is valid by full faithfulness. Locally choose bases compatible with the filtration and with the standard graded Frobenius matrices. This gives the triangular matrix description of [16, Proposition 5.16]. The increasing graded slopes give decreasing diagonal exponents, as required in Lemma D.5. Lemma D.5 replaces it by a meromorphic Frobenius matrix, proving local essential surjectivity. Full faithfulness then descends the object and its filtration.

Finally [16, Proposition 5.22] identifies meromorphic bundles with locally constant generic Newton polygon with their increasing generic slope filtrations. Its construction successively lifts the minimum-slope summand using Proposition 5.9 and Lemma 5.10 of that reference. The pure-slope equivalence of [16, Proposition 5.13] identifies the framed generic graded pieces with the corresponding framed vector bundles. Applying it to $(N_0[1/p], p^{-1}F)$ gives $\mathcal{O} \oplus \mathcal{E}_{n-1}$, as claimed. For the generic analytic isocrystal, [16, Proposition 5.21 and its proof] supplies the canonical slope filtration and v -local standard direct-sum models. Morphisms between distinct slopes vanish, so these local splittings are unique and descend. Taking the direct sum of the frames on the pure summands therefore recovers the full generic frame. $\square$

This is the coheight-one, integrally framed form of [2, Theorem 2.6.3(1)]. We retain the normalized Frobenius and the actual frame actions.

Lemma D.7. *The actual integral frame tower over the Rapoport–Zink height stratum is naturally equivalent to the sheaf of exact sequences*

$$0 \longrightarrow \mathcal{O} \xrightarrow{j} \mathcal{E}_n \xrightarrow{q} \mathcal{E}_{n-1} \longrightarrow 0.$$

On the meromorphic presentation, for $h = (g_n, g_{n-1}, \sigma) \in G_{n,n-1}^0(k)$ and $e \in \mathbb{Q}_p^\times$, the action is

$$j \longmapsto g_n \sigma(j) e^{-1}, \quad q \longmapsto g_{n-1} \sigma(q) g_n^{-1}.$$

For compact frame changes this is the action on the integral tower; rational changes are transported by the gluing equivalence.

Proof. Use Lemma D.1 and apply Lemma D.6 to its meromorphic object. The generic framing supplies the two framed graded pieces, and the special frame identifies the total bundle with $\mathcal{E}_n$. These are exactly the displayed short exact sequence. Conversely, the filtered comparison followed by integral gluing reconstructs the framed shtuka. Lemma D.3 identifies the resulting integral object with the actual tower.

With the left frame convention, a frame from a fixed model to a variable object changes by right composition with the inverse model automorphism. Thus the special frame changes by $a \mapsto \sigma(a) g_n^{-1}$, and the generic frames change by $b \mapsto \sigma(b) \text{diag}(g_{n-1}^{-1}, e^{-1})$. Taking the subobject and the quotient gives the two formulas. Every functor used commutes with base change and the Honda semilinear descent datum. Thus these formulas describe natural action maps, with their multiplication and identity compatibilities. $\square$

D.5 The determinant unit fiber

The unit-fiber description is [2, Theorem 2.6.3(2)]; its fixed-determinant quotient is [2, Proposition 2.7.1(2), (3)]. We verify the normalized integral determinant comparison.

Lemma D.8. *There is an integral determinant comparison $\delta : \det \mathcal{E}_n \xrightarrow{\sim} \det \mathcal{E}_{n-1}$ such that the height-zero infinite-level Lubin–Tate space corresponds, under Lemma D.7, to*

$$\left\{ (j, q) : d_j \in \mathbb{Z}_p^\times \delta \right\},$$

where $d_j(j(1) \wedge \tilde{v}_1 \wedge \cdots \wedge \tilde{v}_{n-1}) = v_1 \wedge \cdots \wedge v_{n-1}$. For every surjection $q : \mathcal{E}_n \twoheadrightarrow \mathcal{E}_{n-1}$ there is a unique frame j_δ with $d_{j_\delta} = \delta$. Forgetting j on the displayed unit fiber is a $\mathbb{Z}_p^\times$ -torsor, and its quotient is the sheaf $\mathcal{Q}$ of all such surjections.

Proof. The determinant map on all framed sequences takes values in the $\mathbb{Q}_p^\times$ -torsor $\text{Isom}(\det \mathcal{E}_n, \det \mathcal{E}_{n-1})$. After forgetting the integral generic frames, its valuation is well-defined on the Rapoport–Zink stratum: their determinants are units. It is locally constant. The height-zero stratum is $\text{LT}_{n,n-1} = \text{Spa } k((u_{n-1}))$, hence is connected. Its determinant valuation is therefore constant. Source quasi-isogenies translate this valuation by reduced-norm valuation and permute the Rapoport–Zink height components transitively; the kernel is $\mathcal{O}_{D_n}^\times$. The height-component description [2, Proposition 2.1.6] consequently shows that the height-zero component is exactly one determinant valuation fiber.

This fiber is the unit coset of a *normalized* integral determinant comparison. We give the construction on perfectoid test rings, so no use of perfect-ring Dieudonné theory over the imperfect field $k((u_{n-1}))$ is involved. For a dimension-one p -divisible group X of height i over a perfect ring A , put

$$L_X = \bigwedge^i M(X), \quad F_{L_X} = p^{1-i} \bigwedge^i F, \quad V_{L_X} = pF_{L_X}^{-1}.$$

The Hodge filtration in covariant integral Dieudonné theory says that $\text{coker } F$ is a projective A -module of rank $i - 1$, killed by p . Locally on $\text{Spec } A$, lift bases of the kernel and image modulo p and lift the changes of basis p -adically. Invertible row and column operations then put the matrix of the linearization of F in the form

$$\begin{pmatrix} 1 & 0 \\ 0 & pC \end{pmatrix}, \quad C \in M_{i-1}(W(A)).$$

Since $p \text{ coker } F = 0$, one has $pW(A)^{i-1} \subseteq pCW(A)^{i-1}$. The ring $W(A)$ is p -torsion-free, so cancellation of p shows that C is surjective, hence invertible. Thus $p^{1-i} \det F$ is a unit in these local bases, and F_{L_X} is an integral isomorphism. Thus (L_X, F_{L_X}, V_{L_X}) is the covariant integral module of a multiplicative-type height-one p -divisible group. Its normalized Frobenius is

$$p^{-1}F_{L_X} = \bigwedge^i (p^{-1}F) = \det \Phi.$$

In particular it corresponds to $\det \mathcal{E}_X$, where $\mathcal{E}_X$ is the bundle associated to the normalized Dieudonné module of X . The unnormalized top exterior power with operator $\bigwedge^i F$ would not be a height-one Dieudonné module when $i \geq 3$.

For the cyclic Honda model,

$$\det F_i = (-1)^{i-1} p^{i-1} \varphi, \quad F_{L_{\Gamma_i}} = (-1)^{i-1} \varphi, \quad \det \Phi_i = (-1)^{i-1} p^{-1} \varphi.$$

Since $n(n-1)$ is even, $\mathbb{F}_{p^2} \subset k$. A unit $u \in W(\mathbb{F}_{p^2})$ satisfying $\varphi(u) = -u$ gives an integral comparison $L_{\Gamma_n} \simeq L_{\Gamma_{n-1}}$. Such units exist, for example by taking the difference of the Teichmüller lifts of two Frobenius-conjugate elements of $\mathbb{F}_{p^2} \setminus \mathbb{F}_p$. More explicitly, for $a \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$, the element $u = [a] - [a^p]$ satisfies $\varphi(u) = -u$ and has nonzero reduction $a - a^p$, so it is a unit for every prime. At $p = 2$, $a \in \mathbb{F}_4 \setminus \mathbb{F}_2$ gives $u \bmod 2 = a + a^2 = 1$.

The equality $\varphi(u) = -u$ is in $W(\mathbb{F}_4)$, where $-1 \neq 1$; the arithmetic sign is retained. All integral comparisons form a $\mathbb{Z}_p^\times$ -torsor. Notice also that the integral étale summand of N_0 has $F = p\varphi$; the factor p it contributes cancels exactly the change from p^{2-n} to p^{1-n} in this determinant normalization. Hence L_{X_0} is identified integrally with $L_{\Gamma_{n-1}}$ by the étale frame.

Here is why a height-zero special marking selects this unit coset. Work v -locally over R° as in Lemma D.2. The determinant construction commutes with crystalline base change. More explicitly, the height-one groups have already been constructed over the perfect ring R° . Their crystals over $A_{\text{cris}}(R^\circ/\varpi)[1/p]$ are the normalized top exterior powers of the original crystals. The top exterior power of the special marking intertwines these normalized operators. Full faithfulness up to isogeny over the f -semiperfect ring R°/ϖ [2, Theorem 2.3.2(1)] therefore realizes it as a quasi-isogeny between the two actual height-one groups there. Its height is the height of the original marking, as may be checked by the determinant valuation over each geometric residue field. A height-zero quasi-isogeny between multiplicative-type height-one groups is an integral isomorphism: after Cartier duality this is the assertion that a rational isomorphism of rank-one $\mathbb{Z}_p$ local systems with valuation zero is integral and invertible. The pair $(R^\circ, (\varpi))$ is henselian. Equivalence of finite étale covers over a henselian pair, at every finite level and then in the inverse limit, uniquely lifts this isomorphism to R° after Cartier duality. This supplies the claimed integral determinant identification without an assertion about deformation theory over an imperfect generic field. Comparing its restriction to R with the determinant of the integral generic frame gives a unit in the automorphism sheaf $\mathbb{Z}_p^\times$. Descending this comparison proves $d_j \in \mathbb{Z}_p^\times \delta$ on the height-zero component. Together with the height-component calculation above, this identifies that component with precisely the asserted unit coset. Every step commutes with base change and the actual Honda frame actions.

For any q , the determinant identity for its kernel is

$$\ker(q) \simeq \det \mathcal{E}_n \otimes (\det \mathcal{E}_{n-1})^{-1}.$$

The comparison δ therefore selects a unique j_δ , compatibly with base change. Since $d_{aj} = a^{-1}d_j$, the frames in the unit fiber are precisely the unit multiples of j_δ . This is the same integral étale-frame torsor as in Lemma D.3. $\square$

D.6 The quotient by the norm-one source group

The following is the quotient identification of [2, Proposition 2.7.1(5)], with its action transported from the normalized integral frames.

Corollary D.9. *For $S_n = \ker(\text{Nrd}_n : \mathcal{O}_{D_n}^\times \rightarrow \mathbb{Z}_p^\times)$ there is an equivalence*

$$[\text{Spd } \widehat{L}_k / S_n] \simeq \text{BC}(-1/(n-1))^\times.$$

The right side classifies extensions of $\mathcal{E}_{n-1}$ by $\mathcal{O}$ that are non-split at every geometric point. Its residual action is the one induced by the normalized frame construction.

Proof. The actual integral frame tower over $\text{Spa } k((u_{n-1}))$ maps to $\text{Spd } \widehat{L}_k$ by forgetting its étale frame. By construction this is the $\mathbb{Z}_p^\times$ -torsor of integral étale frames; Lemma D.3 identifies that torsor in the integral shtuka presentation. Lemmas D.7 and D.8 therefore identify its quotient with the sheaf of all surjections $\mathcal{E}_n \rightarrow \mathcal{E}_{n-1}$. Normalize the kernel frame using the latter lemma. Quotienting the special frame by S_n retains the determinant trivialization and gives extensions

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_{n-1} \longrightarrow 0$$

whose middle bundle is geometrically isomorphic to $\mathcal{E}_n$. Indeed, such bundles are v -locally isomorphic to $\mathcal{E}_n$ by geometric isotriviality of the pure-slope stratum [16, Proposition 5.13 and its proof]. A determinant compatible isomorphism can be chosen because $\mathrm{Nrd}_n : D_n^\times \rightarrow \mathbb{Q}_p^\times$ is surjective. Its kernel already lies in $\mathcal{O}_{D_n}^\times$, since reduced-norm valuation detects the division-algebra valuation. The determinant-compatible frame torsor is therefore exactly an S_n -torsor.

For completeness, the middle bundle is geometrically $\mathcal{E}_n$ if and only if the extension is non-split. On a geometric Fargues–Fontaine curve its Harder–Narasimhan slopes lie between 0 and $1/(n-1)$, and it has rank n and degree 1. A positive slope d/r in lowest terms has $r \leq n$. If $d \geq 2$ then $d/r > 1/(n-1)$; hence $d = 1$ and $r = n-1$ or $r = n$. The only possible polygons are therefore the pure slope $1/n$ and $(1/(n-1), 0)$. In the latter case $\mathcal{E} \simeq \mathcal{E}_{n-1} \oplus \mathcal{O}$. The component of the kernel map in the $\mathcal{O}$ summand must be nonzero: otherwise the quotient map restricted to $\mathcal{E}_{n-1}$ either is an isomorphism, contradicting its nonzero kernel, or is zero, leaving a rank-one map unable to surject onto the rank- $n-1$ bundle $\mathcal{E}_{n-1}$. That component is thus invertible and gives a retraction, so the extension splits. Conversely, a split extension has this second polygon. This proves the equivalence pointwise and hence on families, using the pure-slope locus.

Finally $\mathrm{Ext}^1(\mathcal{E}_{n-1}, \mathcal{O}) = H^1(X_S, \mathcal{E}_{n-1}^\vee)$ is the negative Banach–Colmez sheaf $\mathrm{BC}(-1/(n-1))$, and $\mathrm{Hom}(\mathcal{E}_{n-1}, \mathcal{O}) = 0$. Thus the extension groupoid has no automorphisms fixing its ends, and the everywhere non-split locus is precisely its punctured sheaf. The equivalence transports the actual residual frame action; its source and target actions can then be computed separately as in the main text. $\square$

D.7 Strictness in completed constant-field extension

The residue-cohomology comparison of Lemma D.11 uses the following completed scalar-extension lemma. It verifies the step of [2, Lemma 3.2.8(2)] for these cochain complexes by establishing strictness and constructing a continuous chain contraction.

Lemma D.10. *Let $k \subset k'$ be discrete fields and let $A^\bullet$ be a nonnegative cochain complex of complete metrizable linearly topologized k -vector spaces, with continuous differentials. Suppose their condensifications are solid and the cohomology of this complex in solid k -modules is finite-dimensional and discrete in every degree. Define $A^q \widehat{\otimes}_k k'$ by completing $A^q \otimes_k k'$ for the topology induced by the open k -subspaces of A^q . Then the canonical maps*

$$\mathcal{H}^i(A^\bullet) \otimes_k k' \xrightarrow{\sim} \mathcal{H}^i(A^\bullet \widehat{\otimes}_k k')$$

are isomorphisms of solid k' -modules, with discrete topology on the left.

Proof. Write Z^i , B^i , and $\mathcal{H}^i$ for cycles, boundaries, and solid cohomology. Identify the finite-dimensional discrete cohomology objects with their underlying k -vector spaces. The topological kernel $Z^i \subset A^i$ is closed. The solid quotient $Z^i \rightarrow \mathcal{H}^i$ is a continuous map to the finite-dimensional discrete vector space $\mathcal{H}^i$, and it splits: finite free discrete k -modules are projective in solid k -modules. Thus B^i is closed in Z^i and is complete and metrizable in its induced topology. The differential $A^{i-1} \rightarrow B^i$ is a condensed epimorphism.

This epimorphism is open. Indeed, if an open additive subgroup $U \subset A^{i-1}$ had non-open image, choose $b_j \rightarrow 0$ in B^i outside that image. The map from the compact profinite convergent-sequence space $\{0, 1, 2, \dots\}$ sending 0 to zero and j to b_j lifts continuously after a profinite cover. The compact fiber over 0 maps into the kernel. It has an open neighborhood on which the lift takes values in translates of U by kernel elements. The complementary compact set maps to a closed subset of the sequence space avoiding 0, and hence to only

finitely many isolated points. Therefore all sufficiently late b_j lie in the image of U , a contradiction. Both short exact sequences involving Z^i , B^i , and $\mathcal{H}^i$ are consequently strict exact sequences of complete metrizable linearly topologized vector spaces.

Every open continuous linear map $f : V \rightarrow W$ between complete metrizable linearly topologized k -vector spaces admits a continuous k -linear section. Indeed, choose a decreasing countable neighborhood basis $V = U_0 \supset U_1 \supset \cdots$ of open subspaces and set $\text{Fil}^m W = f(U_m)$. Openness and continuity make $(\text{Fil}^m W)$ a neighborhood basis in W . Put $\text{gr}^m W = \text{Fil}^m W / \text{Fil}^{m+1} W$, with discrete topology, choose linear sections $s_m : \text{gr}^m W \rightarrow \text{Fil}^m W$, and choose linear lifts $\tilde{s}_m : \text{gr}^m W \rightarrow U_m$ with $f\tilde{s}_m = s_m$. Completeness gives a topological linear isomorphism

$$\prod_{m \geq 0} \text{gr}^m W \xrightarrow{\sim} W, \quad (e_m) \mapsto \sum_{m \geq 0} s_m(e_m).$$

Its inverse is obtained by successively taking quotient coordinates and subtracting their chosen representatives. The convergent series $\sum_{m \geq 0} \tilde{s}_m(e_m)$ defines a continuous linear section $s : W \rightarrow V$, since its tails lie in the prescribed U_m . Applying these sections to the two strict exact sequences gives continuous decompositions, with $B^0 = 0$,

$$A^i \simeq B^i \oplus \mathcal{H}^i \oplus B^{i+1}, \quad d(b, h, c) = (c, 0, 0).$$

Let $\iota : \mathcal{H}^\bullet \rightarrow A^\bullet$ and $\rho : A^\bullet \rightarrow \mathcal{H}^\bullet$ be inclusion and projection, where $\mathcal{H}^\bullet$ has zero differential. The continuous chain homotopy $s^i(b, h, c) = (0, 0, b) \in A^{i-1}$ satisfies

$$\rho\iota = \text{id}, \quad ds + sd = \text{id} - \iota\rho.$$

The prescribed completion extends continuous linear maps and preserves finite direct sums, sums, and compositions. These identities therefore remain valid after completed scalar extension. Since $\mathcal{H}^i$ is discrete, $\mathcal{H}^i \widehat{\otimes}_k k' = \mathcal{H}^i \otimes_k k'$.

For a countable neighborhood basis (U_m) in A^i ,

$$A^i \widehat{\otimes}_k k' \simeq \varprojlim_m ((A^i / U_m) \otimes_k k').$$

This is an inverse limit of discrete solid modules, so its condensification is solid. Condensifying the continuous homotopy retains every profinite parameter and proves the comparison in solid modules. The chosen splittings only verify invertibility of the canonical map $[z] \otimes a \mapsto [z \otimes a]$; its naturality comes from scalar extension. $\square$

D.8 Residue cohomology over the required constant fields

The cohomology calculation below is [2, Proposition 3.6.9]. We repeat the descent over the required finite constant fields and verify the canonical completed scalar-extension maps.

Lemma D.11. *Let k_0 be a finite field containing $\mathbb{F}_{p^{n(n-1)}}$. Write $\text{BC}_k(-1/(n-1))^\times$ for the punctured Banach–Colmez space over $\text{Spd } k$. For $k = k_0$ and $k = \overline{\mathbb{F}}_p$, the solid $\mathcal{O}^b$ -cohomology of $\text{BC}_k(-1/(n-1))^\times$ is concentrated in degrees zero and n . Its degree-zero term is k through the constant unit, and its degree- n term is a one-dimensional k -vector space with discrete topology. Moreover, the canonical pullback induces isomorphisms*

$$k' \otimes_{k_0} \mathcal{H}^i R\Gamma(\text{BC}_{k_0}(-1/(n-1))^\times, \mathcal{O}^b) \xrightarrow{\sim} \mathcal{H}^i R\Gamma(\text{BC}_{k'}(-1/(n-1))^\times, \mathcal{O}^b)$$

when k'/k_0 is finite or $k' = \overline{\mathbb{F}}_p$, as solid modules. These maps retain the geometric group actions and coefficient units.

Proof. First work over k_0 , without reducing its constant field. The normalized integral-frame comparison of this appendix gives

$$\mathrm{BC}_{k_0}(-1/(n-1))^\times \simeq [\mathrm{Spd} \widehat{L}_{k_0}/S_n].$$

Here the cover is affinoid perfectoid by Lemma C.1; its action nerve has terms $\mathrm{Spd} \widehat{L}_{k_0} \times S_n^j$. In particular the quotient is qcqs. The same field lemma identifies $\widehat{L}_{k_0}$, over k_0 , with the completed perfection of a Laurent series field. Thus [2, Corollary 3.6.8] applies with $q = |k_0|$, including its solid enhancement and natural pullback. Taking the commuting S_n -invariants gives

$$R\Gamma(\mathrm{BC}_{k_0}(-1/(n-1))^\times, \mathcal{O}^\flat) \simeq R\Gamma(\mathrm{BC}_{k_0}(-1/(n-1))^\times \times_{\mathrm{Spd} k_0} \mathrm{Spd} C, \mathcal{O}^\flat)^{h\varphi_q},$$

where C is a complete algebraically closed untilt with the specified embedding of k_0 into $C^\flat$. Choose the unramified degree- $n-1$ extension $F/\mathbb{Q}_p$. Proposition 3.1.2 of [2] identifies the base-changed space with $\mathbb{H}_{F,C}^1/F$. Proposition 3.5.6 of loc. cit. computes its cohomology in degrees zero and n , with a copy of $C^\flat$ in degree zero and a one-dimensional $C^\flat$ -line in the other degree. This calculation is valid for every prime; the additive translation lattice has rank $n-1$, with finite characteristic- p coefficients. The maps in degree zero are the specified constant units.

The Frobenius action on the upper line is semilinear. Over the algebraically closed field $C^\flat$, a basis can be rescaled to be fixed by φ_q : the required equation is $x^{q-1} = c$ for a nonzero $c \in C^\flat$. Artin–Schreier surjectivity then shows that $\varphi_q - 1$ is surjective on each of the two coefficient lines, with kernel a one-dimensional k_0 -space. These are assertions about solid modules: Artin–Schreier has local continuous sections, and a continuous map from a profinite set has compact image, so finitely many such sections and a finite clopen refinement provide the required parameterwise lifts. The two-term Frobenius fixed-point sequence proves the asserted finite-field calculation. This repeats the proof of [2, Proposition 3.6.9] over a constant field on which both Honda endomorphism rings and the actual frame tower are defined. No comparison over the smaller field $\mathbb{F}_{p^{n-1}}$ is needed.

For constant extension, use the preceding action nerve. Its normalized coefficient cochains are closed subspaces of $C(S_n^j, \widehat{L}_{k_0})$, with the complete sup-norm topology; they are complete metrizable linearly topologized k_0 -spaces. They are solid, the differentials are continuous, and the finite-field calculation has just established finite-dimensional discrete solid cohomology in every degree. Base change of each affinoid perfectoid nerve term extends its ring by completed tensor product [2, Lemma 3.2.8(1)]. To identify its continuous functions, let S be profinite and give $C(S, V)$ the uniform topology for a complete linearly topologized k_0 -space V . Write $\mathrm{LC}(S, M)$ for locally constant functions with values in a discrete k_0 -space M , following [2, Theorem 3.4.1]. For each open subspace $V_0 \subset V$,

$$C(S, V)/C(S, V_0) \simeq \mathrm{LC}(S, V/V_0).$$

A locally constant function lifts by choosing representatives on a finite clopen partition. The finite-partition description of $\mathrm{LC}(S, -)$ also makes it commute with scalar extension. Taking inverse limits over V_0 , with these quotient topologies defining the completion, gives the canonical isomorphism

$$C(S, V) \widehat{\otimes}_{k_0} k' \simeq C(S, V \widehat{\otimes}_{k_0} k').$$

Apply this with $V = \widehat{L}_{k_0}$ and $S = S_n^j$, or $S = U \times S_n^j$ for an additional profinite parameter U . By the base-change compatibility of Corollary D.9 and Lemma C.2, this is the action nerve of the actual base-changed frame field and presents $\mathrm{BC}_{k'}(-1/(n-1))^\times$. Its action

maps and constant units are the base changes of the specified ones. This also commutes with the finite kernels defining normalized cochains: the degeneracy splittings are continuous and are preserved by completed scalar extension. Lemma D.10 therefore gives the claimed canonical isomorphisms, including the extra profinite parameters. All identifications may also be made before normalization.

The comparison morphisms are induced by the original base-change and coefficient-unit maps. Their proof of invertibility uses choices of bases, but the morphisms themselves use none. Consequently they commute with every geometric action and with the constant units. Taking $k' = \overline{\mathbb{F}}_p$ proves the remaining assertions. $\square$

E The geometric comparisons used in Cartan detection

Lemma 4.5 uses the two tower comparisons of Lemma E.1 and the fixed-point error estimate of Lemma E.4. The comparisons are low-degree forms of the Lubin–Tate and Drinfeld comparisons of Barthel, Schlank, Stapleton, and Weinstein [3, Theorems 3.9.4, 6.1.4, and 6.2.6]. The supporting proofs retain the coefficient units and continuous diagram compatibilities used in that detection.

Pro-étale cohomology with coefficients $\widehat{\mathcal{O}}^+$ uses the solid enhancement obtained by evaluation on profinite parameters as in [3, §3.6]. For a coefficient ring A , the notation $A[\epsilon]$ below means the two-term complex $A \oplus A[-1]$, with ϵ in cohomological degree one.

Lemma E.1. *There are continuous equivariant maps*

$$\alpha_{\text{LT}} : R_n[\epsilon_{\text{LT}}] \longrightarrow R\Gamma_{\text{cond}}((\text{LT}_{n,\eta})_{\text{proét}}, \widehat{\mathcal{O}}^+) \quad \text{in } D(\text{Solid}_{G_n})$$

and

$$\alpha_{\text{Dr}} : \mathbb{Z}_p[\epsilon_{\text{Dr}}] \longrightarrow R\Gamma_{\text{cond}}((\mathbb{H}^{n-1})_{\text{proét}}, \widehat{\mathcal{O}}^+) \quad \text{in } D(\text{Solid}_{\text{GL}_n(\mathbb{Z}_p)}).$$

Their degree-zero restrictions are the formal-to-analytic coefficient units, and the action on each degree-one generator is trivial. Let Q_{LT} and Q_{Dr} be their cofibers. Both lie in $D^{\geq 1}$, and for each $1 \leq b \leq 3$ there are integers $N_{\text{LT},b}$ and $N_{\text{Dr},b}$ such that

$$p^{N_{\text{LT},b}} \mathcal{H}^b(Q_{\text{LT}}) = 0, \quad p^{N_{\text{Dr},b}} \mathcal{H}^b(Q_{\text{Dr}}) = 0.$$

The generators may be chosen independently. Within either construction, the comparison maps commute with restrictions, profinite parameters, and the action groupoid.

The affine input is Lemma F.1, whose projection step uses Lemma E.2. For Drinfeld space, formal structure-sheaf cohomology and the stratum calculation identify the degree-zero unit with $\mathbb{Z}_p$. The Lubin–Tate closed-ball exhaustion identifies its input with R_n . Uniform local bounds then give the asserted cofiber bounds.

E.1 Flat coefficient factors

Completed tensor products in this appendix are derived tensor products followed by derived completion. The factors used in the affine Hodge comparison are rings of formal functions and finite projective modules of logarithmic differentials. They are flat complete modules in degree zero. We prove the flat degree-zero projection formula needed in the affine comparison. The counterexample after the proof explains this restriction relative to [3, Lemma 3.4.10].

Lemma E.2. *Let $\mathcal{O}$ be a complete discrete valuation ring with uniformizer π , and let P be a profinite group. Let M, N be flat, separated, π -adically complete $\mathcal{O}$ -modules, whose*

quotients modulo π^r are discrete. Give M the trivial action and N a continuous $\mathcal{O}$ -linear P -action. Then

$$M \hat{\otimes}_{\mathcal{O}} R\Gamma_{\text{cts}}(P, N) \longrightarrow R\Gamma_{\text{cts}}(P, M \hat{\otimes}_{\mathcal{O}} N)$$

is an equivalence of solid complexes.

Proof. Both sides are derived π -complete. Reduction modulo π commutes with derived invariants, since it is the cofiber of multiplication by π and invariants preserve finite fiber sequences. Derived Nakayama reduces the assertion to the projection map for $M/\pi M$ and $N/\pi N$.

Choose a basis of the residue-field vector space $M/\pi M$. In continuous cochain degree q , the reduced map is

$$(M/\pi M) \otimes_{\mathcal{O}/\pi} C_{\text{cts}}(P^q, N/\pi N) \longrightarrow C_{\text{cts}}(P^q, (M/\pi M) \otimes_{\mathcal{O}/\pi} (N/\pi N)).$$

A continuous function from the compact space P^q to this discrete target has finite image. Its values involve finitely many basis vectors, and their coordinate functions are continuous. This gives an isomorphism in every cochain degree. Tensoring over the residue field is exact.

For an extremally disconnected profinite test set S , use $S \times P^q$ in the same argument. Compactness again supplies one finite set of basis vectors. The isomorphisms respect maps of S and all cochain differentials, so they identify the internal solid complexes. Derived Nakayama proves the result. $\square$

The bounded-above formulation of [3, Lemma 3.4.10] is too general. For example, take trivial actions, $P = C_p$, $N = \mathbb{F}_p$, and $M = \bigoplus_{j \geq 0} \mathbb{Z}_p[j]$, with the convention $\mathcal{H}^a(C[j]) = \mathcal{H}^{a+j}(C)$. These are bounded-above derived p -complete complexes. The cyclic resolution has zero differential after applying $\text{Hom}_{\mathbb{Z}_p[C_p]}(-, \mathbb{F}_p)$. Tensor totalization and Hom totalization therefore give

$$\mathcal{H}^0(M \hat{\otimes} R\Gamma_{\text{cts}}(C_p, \mathbb{F}_p))(*) = \bigoplus_{j \geq 0} \mathbb{F}_p, \quad \mathcal{H}^0(R\Gamma_{\text{cts}}(C_p, M \hat{\otimes} \mathbb{F}_p))(*) = \prod_{j \geq 0} \mathbb{F}_p.$$

The projection map is the inclusion of the direct sum. Lemma E.2 supplies the projection formula for the coefficient factors used here.

E.2 The formal unit on Drinfeld space

Let $\mathfrak{H}^{n-1}$ be the standard semistable formal model of $\mathbb{H}^{n-1}$ over $\mathbb{Z}_p$, obtained by taking the p -adic completion of the weak formal model in [8, §§6.1–6.2]. It is flat and locally Noetherian, with strict normal crossings special fiber. Its dual complex is the Bruhat–Tits building of $\text{PGL}_n(\mathbb{Q}_p)$. Each nonempty closed stratum is a finite product of varieties obtained from projective spaces over $\mathbb{F}_p$ by successive blowups along smooth transforms of rational linear subspaces of positive codimension.

The model is separated. Its algebraic stages are open subschemes of finite blowups of projective space over $\mathbb{Z}_p$, and form an increasing union by open immersions. On the product of two stages, the diagonal is the restriction of the closed diagonal of a stage containing both. The union and its formal completion are therefore separated.

The following is [3, Theorem 6.2.1], restricted to $\text{GL}_n(\mathbb{Z}_p)$. We give a closed-stratum proof using smooth blowup centers of positive codimension.

Lemma E.3. *For the p -adically topologized formal structure sheaf, the unit*

$$\mathbb{Z}_p \longrightarrow R\Gamma_{\text{Zar}}(\mathfrak{H}^{n-1}, \mathcal{O}_{\text{cond}})$$

is an equivalence of solid complexes with continuous $\text{GL}_n(\mathbb{Z}_p)$ -action. The action on $\mathbb{Z}_p$ is trivial.

Proof. Let Y be the special fiber, with irreducible components Y_v , and write $Y_I = \bigcap_{v \in I} Y_v$ for a nonempty closed stratum. We first prove

$$\mathbb{F}_p \xrightarrow{\sim} R\Gamma_{\text{Zar}}(Y_I, \mathcal{O}_{Y_I}). \quad (\text{E.1})$$

If $b : X' \rightarrow X$ is the blowup of a smooth variety along a smooth closed center of positive codimension, then $\mathcal{O}_X \rightarrow Rb_*\mathcal{O}_{X'}$ is an equivalence. By the local coordinates for smooth closed pairs [7, Tag 0FUE], it suffices to check the blowup of $\mathbb{A}^a \times \mathbb{A}^c$ along $\mathbb{A}^a \times \{0\}$, where $c \geq 1$. The blowup of $\mathbb{A}^c$ at the origin is the total space of $\mathcal{O}_{\mathbb{P}^{c-1}}(-1)$. Its projection to $\mathbb{P}^{c-1}$ has direct image $\bigoplus_{r \geq 0} \mathcal{O}_{\mathbb{P}^{c-1}}(r)$. Projective-space cohomology gives the polynomial ring in degree zero and zero in higher degrees. A finite affine Čech calculation commutes with flat étale base change. This proves the claim locally at the center, and the blowup is an isomorphism off the center. Induction on the blowups and coherent Künneth over $\mathbb{F}_p$ now prove (E.1).

For a discrete $\mathbb{F}_p$ -algebra A and a profinite S ,

$$C_{\text{cts}}(S, A) = C_{\text{cts}}(S, \mathbb{F}_p) \otimes_{\mathbb{F}_p} A.$$

Both expressions use finite clopen partitions of S . Each stratum is projective and separated, so a finite affine cover with affine intersections computes its coherent cohomology. Tensor that calculation with $C_{\text{cts}}(S, \mathbb{F}_p)$. Affine acyclicity, also in the condensed calculation, follows from [3, Lemma 3.5.1]. Thus (E.1) holds as an equivalence of solid complexes, with its unit and its restriction maps.

For the closed immersions $i_I : Y_I \rightarrow Y$, strict normal crossings give the resolution

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_Y \longrightarrow \prod_{|I|=1} (i_I)_*\mathcal{O}_{Y_I} \longrightarrow \prod_{|I|=2} (i_I)_*\mathcal{O}_{Y_I} \longrightarrow \cdots \\ \longrightarrow \prod_{|I|=n} (i_I)_*\mathcal{O}_{Y_I} \longrightarrow 0, \end{aligned} \quad (\text{E.2})$$

with alternating restriction maps. Exactness is étale local. In a normal-crossings chart $\mathbb{F}_p[x_1, \dots, x_n]/(x_1 \cdots x_r)$, a nonzero monomial has a nonempty set of zero exponents among $x_1, \dots, x_r$. Its summands form the augmented cochain complex of the simplex on this set, which is exact. Decomposition by monomials proves exactness on the chart; flat étale pullback proves the sheaf assertion. Tensoring with $C_{\text{cts}}(S, \mathbb{F}_p)$ proves the condensed version.

The components form a locally finite family. Near each point, the products in (E.2) are finite, and hence also derived products. Restriction to an open preserves derived products: its left adjoint, extension by zero, is exact. Thus these are derived products globally. Closed-immersion pushforward is exact, and derived sections preserve products. The stratum units consequently identify $R\Gamma_{\text{Zar}}(Y, \mathcal{O}_{Y, \text{cond}})$ with the simplicial cochains of the building, with solid coefficients $\mathbb{F}_p$.

The building is contractible, so its augmented simplicial chain complex over $\mathbb{F}_p$ is split exact. A chain contraction sends every basis simplex to a finite sum. Its dual therefore defines a contraction on the products in the solid cochain complex. This proves that the unit $\mathbb{F}_p \rightarrow R\Gamma_{\text{Zar}}(Y, \mathcal{O}_{Y, \text{cond}})$ is an equivalence on every profinite test set.

The formal structure sheaf is p -torsion-free and derived p -complete on formal affines. Derived sections preserve completeness and commute with reduction modulo p . The cofiber of the formal unit is therefore derived p -complete with zero reduction modulo p . Derived Nakayama proves the desired equivalence. The unit commutes with the formal group action; forgetting that action detects equivalences. $\square$

Separatedness gives an affine cover with affine finite intersections: the intersection of two affine opens is the inverse image of the closed diagonal in their affine fiber product. These intersections are still semistable. The equality of formal functions and bounded analytic functions in [3, Lemma 5.2.3], together with affine acyclicity, identifies the formal Čech complex with the bounded-function Čech complex. The affine maps of Appendix F thus descend by the construction of [3, Theorem 5.6.8]. Their local cofibers lie in $D^{\geq 1}$ and have uniform bounds in degrees one through three. The Čech spectral sequence has finitely many terms in each of these total degrees, so its cofiber has the same lower bound and a bounded p -torsion exponent in each of them. Lemma E.3 identifies the descended source with $\mathbb{Z}_p[\epsilon_{\text{Dr}}]$. All maps are obtained from the coefficient unit and the chosen base-field map, and hence respect the action groupoid. This proves the Drinfeld assertion of Lemma E.1.

E.3 The Lubin–Tate open ball

The Lubin–Tate open ball has dimension $n - 1$ over $W(\overline{\mathbb{F}}_p)[1/p]$. Choose increasing primes $\ell_m \neq p$ tending to infinity. The closed balls

$$X_m = \{ |T_i| \leq |p|^{1/\ell_m}, 1 \leq i \leq n - 1 \}$$

exhaust the Lubin–Tate open ball. They are preserved by the stabilizer action: each $g(T_i)$ lies in the maximal ideal $(p, T_1, \dots, T_{n-1})$ of $W(\overline{\mathbb{F}}_p)[[T_1, \dots, T_{n-1}]]$. Since $|p|^{1/\ell_m} > |p|$, the norm of $g(T_i)$ on X_m is at most $\max(|p|, |p|^{1/\ell_m}) = |p|^{1/\ell_m}$. Applying the same estimate to g^{-1} shows that $g(X_m) = X_m$. After adjoining p^{1/ℓ_m} to $W(\overline{\mathbb{F}}_p)[1/p]$, each X_m has the smooth model obtained by using the coordinates $T_i/p^{1/\ell_m}$. The tame uniformity of Lemma F.1 gives compatible maps

$$\mathcal{O}^+(X_m)[\epsilon_{\text{LT}}] \longrightarrow R\Gamma_{\text{cond}}((X_m)_{\text{proét}}, \widehat{\mathcal{O}}^+)$$

whose cofibers Q_m lie in $D^{\geq 1}$. There are exponents N_b , independent of m , killing $\mathcal{H}^b(Q_m)$ for $1 \leq b \leq 3$. The generator is fixed within this exhaustion.

The Mittag–Leffler calculation of [3, Lemmas 6.1.2–6.1.3] gives

$$R\varprojlim_m \mathcal{O}^+(X_m) \simeq W(\overline{\mathbb{F}}_p)[[T_1, \dots, T_{n-1}]] \quad (\text{E.3})$$

with the $(p, T_1, \dots, T_{n-1})$ -adic topology on the right. To see why the ordinary calculation supplies the solid one, fix a radius $r < 1$ and an integer $j \geq 1$. For all sufficiently large radii $s < 1$ and all $j' \geq j$, [3, Lemma 6.1.3] identifies the image modulo p^j with a fixed finite-degree span of integral monomials. A continuous map from a profinite set to a discrete quotient has finite image; lifting its values on a finite clopen partition proves the same stable-image assertion after evaluation. The two-index system of radii and powers of p has a countable cofinal subsystem. Its Mittag–Leffler condition removes higher inverse limits. Taking the ordinary limit gives precisely the formal power series with their displayed topology.

Descent for the increasing cover and exactness of derived limits identify the final comparison cofiber with $R\varprojlim_m Q_m$. Products are exact in solid modules, so

$$0 \longrightarrow \varprojlim_m^1 \mathcal{H}^{b-1}(Q_m) \longrightarrow \mathcal{H}^b(R\varprojlim_m Q_m) \longrightarrow \varprojlim_m \mathcal{H}^b(Q_m) \longrightarrow 0.$$

The outside terms are killed by $p^{N_{b-1}}$ and p^{N_b} , respectively, with $N_0 = 0$ since $Q_m \geq 1$. This gives the asserted low-degree bounds and retains the lower bound $D^{\geq 1}$. The structure unit in (E.3) is the Lubin–Tate coefficient unit. The constructions are natural on the entire exhaustion and on its action groupoid, including semilinear arithmetic automorphisms. They therefore give α_{LT} in $D(\text{Solid}_{G_n})$. This completes the proof of Lemma E.1.

E.4 Low degrees after continuous invariants

We record an elementary estimate from the finite Postnikov filtration.

Lemma E.4. *Let P be a profinite group, and let $Q \in D^{\geq 1}(\text{Solid}_P)$. Suppose $p^{N_b}\mathcal{H}^b(Q) = 0$ for $1 \leq b \leq 3$. Then*

$$p^{N_1+N_2}\mathcal{H}^2(R\Gamma_{\text{cts}}(P, Q)) = 0, \quad p^{N_1+N_2+N_3}\mathcal{H}^3(R\Gamma_{\text{cts}}(P, Q)) = 0.$$

Proof. Continuous derived invariants preserve $D^{\geq 0}$. Consequently their cohomology through degree r depends only on $\tau_{\leq r}Q$. The finite Postnikov filtration of that truncation has pieces $\mathcal{H}^b(Q)[-b]$, for $1 \leq b \leq r$. Multiplication by p^{N_b} is zero on the coefficient module $\mathcal{H}^b(Q)$, so it remains zero after derived invariants. The long exact sequences of this finite filtration show that $p^{N_1+\dots+N_r}$ kills its degree- r cohomology. Apply this for $r = 2, 3$. $\square$

The long exact cofiber sequence now makes the Lubin–Tate comparison injective on $H^3[1/p]$ after G_n -invariants, and the Drinfeld comparison an isomorphism on $H^3[1/p]$ after $\text{GL}_n(\mathbb{Z}_p)$ -invariants. After restriction to the block $\text{SL}_2(\mathbb{Z}_p)$ used in Cartan detection, the same argument gives the needed injection. These conclusions use the bounded-below cofibers, through their finite low-degree Postnikov filtrations.

E.5 Constant coefficients on the quotient stacks

For each pair

$$(\text{LT}_{n,\eta}, G_n), \quad (\mathbb{H}^{n-1}, \text{GL}_n(\mathbb{Z}_p)), \quad (\mathbb{H}^{n-1}, \text{SL}_2(\mathbb{Z}_p)),$$

where $\text{SL}_2(\mathbb{Z}_p)$ acts through the last block as in Lemma 4.5, the coefficient-unit map from constant cohomology is compatible with

$$R\Gamma_v([X^\diamond/P], \mathbb{Z}_p) \simeq R\Gamma_{\text{cond}}([X/P]_{\text{proét}}, \mathbb{Z}_p). \quad (\text{E.4})$$

Here the right side means the continuous quotient-nerve cohomology, and $\mathbb{Z}_p$ on either side means $R\varprojlim_r \mathbb{Z}/p^r$. The comparison also applies at height $n-1$ to the target unit used in the target coefficient evaluation (4.19).

Fix r . For a locally spatial diamond, pullback from the étale site preserves cohomology of the finite constant sheaf $\mathbb{Z}/p^r$ on the quasi-pro-étale and v -sites [13, Propositions 14.7, 14.8, and 14.10]. These results include p -primary coefficients. The étale sites of a rigid space and its diamond agree [13, Lemma 15.6]. The comparison with the rigid pro-étale site is [14, Lemma 3.16 and Corollary 3.17(i)], with the topology specified in its erratum.

To include continuous parameters, let U be a quasicompact quasiseparated rigid chart and write a profinite set as $S = \varprojlim_i S_i$, with S_i finite clopen quotients. The pro-object $U \times S = \varprojlim_i (U \times S_i)$ belongs to the rigid pro-étale site: its transition maps are finite étale and surjective. By [14, Lemma 3.16] and [13, Proposition 14.9], the preceding comparisons identify

$$\begin{aligned} R\Gamma((U \times S)_{\text{proét}}, \mathbb{Z}/p^r) &\simeq \text{colim}_i R\Gamma_{\text{ét}}(U, \mathbb{Z}/p^r)^{S_i} \\ &\simeq R\Gamma_v(U^\diamond \times S, \mathbb{Z}/p^r). \end{aligned}$$

These equivalences respect restriction to rigid charts and maps of profinite parameters. Descent over the charts gives the comparison on X .

The continuous action is formed on the modern side. The actual map $P \times X^\diamond \rightarrow X^\diamond$ defines the simplicial diamond $X^\diamond \times P^\bullet$. Transport its full cochain diagram along the parameterized comparisons, including an additional parameter S . This supplies the rigid

cochain model with its face maps, degeneracy maps, and composition coherences. It defines the continuous quotient-nerve cohomology on the right of (E.4).

Descent and totalization give the comparison for $\mathbb{Z}/p^r$ on the quotient. Taking $R\varprojlim_r$ gives (E.4), since both totalization and derived inverse limit are limits. On rigid charts this derived coefficient is the usual p -adic constant sheaf by [14, Proposition 8.2]. At finite level, the coefficient units on both sides come from the same étale map $\mathbb{Z}/p^r \rightarrow \mathcal{O}_{X,\text{ét}}^+/p^r$. The integral comparison in Appendix F.1 identifies their images with all profinite parameters and the continuous action nerve. Taking the limit therefore respects $\mathbb{Z}_p \rightarrow \widehat{\mathcal{O}}^+$.

Finally, the actual common tower in [3, Theorem 3.9.1] gives a bisimplicial refinement of the Lubin–Tate and Drinfeld quotient presentations. The constant-coefficient comparison and its units agree on that refinement and after restriction to the last $\mathrm{SL}_2(\mathbb{Z}_p)$ block. Thus the constant Cartan class defined by the specified source torsor has the same completed-structure-sheaf image in the two presentations. This is the equality used in Lemma 4.5.

F Continuous equivariant integral comparison

This appendix establishes the affine comparison of Lemma F.1, used to prove the two tower comparisons of Lemma E.1.

Let p be any prime, and put

$$e_p = \left\lceil \frac{3}{p-1} \right\rceil.$$

Thus p^{e_p} is 8 for $p = 2$, 9 for $p = 3$, and p for $p \geq 5$. Let K be a complete discretely valued extension of $\mathbb{Q}_p$ with perfect residue field, and let C be the completion of an algebraic closure of K . For an affine semistable formal scheme $\mathfrak{U} = \mathrm{Spf} A$ over $\mathcal{O}_K$, write U_K for its rigid generic fiber. Choose the cyclotomic $\mathbb{Z}_p$ -extension of K by taking the maximal $\mathbb{Z}_p$ -quotient of the Galois group of the extension adjoining all p -power roots of unity [3, §4.2]. For $K = \mathbb{Q}_2$, this removes the torsion factor $\{\pm 1\}$ from $\mathbb{Z}_2^\times = \{\pm 1\} \times (1 + 4\mathbb{Z}_2)$.

For the maximal cyclotomic $\mathbb{Z}_2$ -quotient over an unramified field, the upper jumps are $u_0 = -1$ and $u_i = i + 1$ for $i \geq 1$. Indeed, over $\mathbb{Q}_2$ its degree- 2^n layer is

$$K_n = \mathbb{Q}_2(\zeta_{2^{n+2}} + \zeta_{2^{n+2}}^{-1}).$$

Writing $t_n = \zeta_{2^{n+2}} + \zeta_{2^{n+2}}^{-1}$ and $t_0 = 0$, the relation $t_n^2 = 2 + t_{n-1}$ is Eisenstein at each step. Thus t_n is a uniformizer and

$$v_{K_n}(\mathfrak{D}_{K_n/K_{n-1}}) = v_{K_n}(2t_n) = 2^n + 1.$$

The different formula in [3, Lemma 4.2.2] gives the stated jumps; unramified base change preserves this calculation. For a finite tame extension L of an unramified field, positive upper jumps scale by e_L , and hence the relative different exponent in its base-changed degree- 2^n tower is $e_L 2^n + 1$. In particular it is at least $e_L(2^n - 1) + 1$, the sufficiently ramified bound of [3, (4.2.4)]. We use this normalization when $p = 2$.

Choose the base-field map

$$\alpha_{0,K} : \mathcal{O}_K[\epsilon] \longrightarrow R\Gamma_{\mathrm{cond}}((\mathrm{Spa} K)_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}^+)$$

constructed by cyclotomic inflation in [3, (4.3.7) and (5.3.1)]. Pullback to U_K , followed by multiplication with the formal coefficient unit, defines

$$\alpha_{\mathfrak{U}} : \underline{A}[\epsilon] \longrightarrow R\Gamma_{\mathrm{cond}}((U_K)_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}^+).$$

Here A has its p -adic topology.

The following is the low-degree form of the semistable integral comparison of [3, Theorem 5.3.4], with tame descent as in [3, Lemmas 5.6.1–5.6.2]. We retain continuous chart and action compatibilities and make the dimension bounds and small-prime cyclotomic normalization explicit.

Lemma F.1. *The cofiber $Q_{\mathfrak{U}}$ of $\alpha_{\mathfrak{U}}$ lies in $D^{\geq 1}$, and there is an integer $N(K, d)$ such that*

$$p^{N(K, d)} \mathcal{H}^b(Q_{\mathfrak{U}}) = 0 \quad (1 \leq b \leq 3),$$

where d is the relative dimension. The exponent can be chosen uniformly after finite tame extensions of K . The maps retain their coefficient units and are natural in the continuous semilinear chart, cover, and exhaustion diagrams used for Lubin–Tate and Drinfeld space.

We prove the lemma by factoring the coefficient unit through the continuous logarithmic Hodge comparison of Lemma F.2. The positive Hodge pieces are controlled by the field estimates for $\mathcal{O}_C\{-j\}$ with $1 \leq j \leq 3$. The error between the Hodge complex and the pro-étale complex is controlled by the truncated cofiber bound of Lemma F.3. Its construction in the equivariant diagram category keeps the bound after sections, invariants, and inverse limits.

F.1 The ringed diagram with continuous parameters

For a rigid chart X , write $X_{\text{proét}}^{\text{old}}$ for the rigid pro-étale site of [14], with its corrected topology. Ordinary derived sections are written $R\Gamma(X_{\text{proét}}^{\text{old}}, -)$. Their continuous solid enhancement, denoted $R\Gamma_{\text{cond}}(X_{\text{proét}}, -)$ in the comparisons, is constructed by finite clopen parameters. The diamond quasi-pro-étale model then supplies its full continuous action. For an ordinary complex D , define its derived p -complete condensation by

$$\text{Cond}_p(D) = R\varprojlim_r \gamma^*(D \otimes^L \mathbb{Z}/p^r),$$

where γ^* is discrete condensation. The resulting derived p -complete object is solid by [3, Lemma 3.4.7].

On the original formal étale site, with condensed coefficients, set

$$A_C = A \hat{\otimes}_{\mathcal{O}_K} \mathcal{O}_C, \quad U_C = (\mathfrak{U}_{\mathcal{O}_C})_{\eta}.$$

The coefficient sheaf $\mathcal{A}$ is specified by

$$\mathcal{A}(\mathfrak{U}, S) = C_{\text{cts}}(S, A_C) = \varprojlim_r C_{\text{cts}}(S, A_C/p^r)$$

for profinite S . The ring A_C is p -torsion-free, by semistability and flatness of $\mathcal{O}_C$ over $\mathcal{O}_K$. It has its completed p -adic topology. Modulo p^r the formula satisfies étale descent; taking the limit gives the displayed ringed sheaf. The semilinear action of $\Gamma_K = \text{Gal}(\overline{K}/K)$ is continuous: modulo p^r each coefficient is a finite sum of elementary tensors, with continuous orbit maps.

The ordinary derived direct image is taken along the composite

$$(U_C)_{\text{proét}}^{\text{old}} \longrightarrow (\mathfrak{U}_{\mathcal{O}_C})_{\text{ét}} \longrightarrow \mathfrak{U}_{\text{ét}}.$$

For an affine étale chart $\mathfrak{V} \rightarrow \mathfrak{U}$, write $V_C = (\mathfrak{V}_{\mathcal{O}_C})_{\eta}$. For a rigid affinoid X , put

$$D_X = R\Gamma(X_{\text{proét}}^{\text{old}}, \hat{\mathcal{O}}^+), \quad F_r = \mathcal{O}_{X, \text{ét}}^+/p^r.$$

By [14, Lemma 4.2(iii)], the quotient *sheaf* $\widehat{\mathcal{O}}^+/p^r$ is the pullback of F_r to the rigid pro-étale site. For an extremally disconnected profinite set $S = \varprojlim_i S_i$, Lemma 3.16 of loc. cit. therefore gives

$$\begin{aligned} R\Gamma((X \times S)_{\text{proét}}^{\text{old}}, \widehat{\mathcal{O}}^+/p^r) &\simeq \operatorname{colim}_i R\Gamma_{\text{ét}}(X, F_r)^{S_i} \\ &\simeq \operatorname{colim}_i (D_X \otimes^L \mathbb{Z}/p^r)^{S_i}. \end{aligned}$$

The second equivalence uses p -torsion-freeness of $\widehat{\mathcal{O}}^+$; derived reduction commutes with sections because $\mathbb{Z}/p^r$ is perfect over $\mathbb{Z}$. The sheaf $\widehat{\mathcal{O}}^+$ on this site is derived p -complete by [9, Remark 5.5]. Taking derived inverse limits thus identifies the ordinary parameter functor

$$S \longmapsto R\Gamma((X \times S)_{\text{proét}}^{\text{old}}, \widehat{\mathcal{O}}^+)$$

with $S \mapsto \operatorname{Cond}_p(D_X)(S)$. This gives its solid enhancement. Evaluation on extremally disconnected parameters is exact, so it commutes with the derived limit; other profinite parameters are obtained by descent from these generators.

Apply this construction on every affine étale chart: $\mathcal{C}_{\text{proét}}(\mathfrak{V}) = \operatorname{Cond}_p(D_{V_C})$. The finite-level comparisons respect restriction and the formal-to-generic-fiber coefficient units, and hence give an $\mathcal{A}$ -module sheaf on the original formal site. It lies in $D^{\geq 0}$, and its zeroth cohomology is p -torsion-free.

The modern integral coefficient model is linked to this one through the same étale sheaves F_r . For a locally spatial diamond over $\operatorname{Spd} \mathbb{Q}_p$, the construction in [11, §2.6, after Theorem 2.6.2] gives an étale sheaf $\mathcal{O}^+/p$ whose pullback to quasi-pro-étale perfectoids is their integral structure sheaf modulo p . For $X^\diamond$, it identifies this étale sheaf with $\mathcal{O}_{X, \text{ét}}^+/p$. The same identification holds modulo p^r : use the natural coefficient map and the short exact sequences

$$0 \longrightarrow \mathcal{O}^+/p^{r-1} \xrightarrow{p} \mathcal{O}^+/p^r \longrightarrow \mathcal{O}^+/p \longrightarrow 0.$$

Inductively, the essential image of the derived étale pullback contains the middle term because it is closed under extensions by [13, Proposition 14.10]. The natural coefficient map identifies it with the pullback of F_r , by the same short exact sequences and the already established comparisons at $r-1$ and 1. Consequently

$$\begin{aligned} R\Gamma(X_{\text{proét}}^{\text{old}}, \widehat{\mathcal{O}}^+/p^r) &\simeq R\Gamma_{\text{ét}}(X, F_r) \\ &\simeq R\Gamma_{\text{qproét}}(X^\diamond, \mathcal{O}^+/p^r). \end{aligned}$$

All these comparisons allow p -primary coefficients.

On the modern site, choose strictly totally disconnected perfectoid charts with extremally disconnected component space. Their étale section functor is exact. For such a chart T with untilt $\operatorname{Spa}(R, R^+)$, the preceding coefficient identification and [13, Proposition 14.8] therefore give

$$R\Gamma_{\text{qproét}}(T, \mathcal{O}^+/p^r) \simeq (R^+/p^r)[0].$$

The quotient tower is surjective, so its derived limit is R^+ , with no higher limit term. Descent proves that $\mathcal{O}^+$ on the modern quasi-pro-étale site is derived p -complete. This use of modern perfectoid covers is internal to that site. Combining the finite-level comparison with [13, Proposition 14.9] for the finite clopen quotients and then taking the derived limit gives

$$R\Gamma_{\text{qproét}}(X^\diamond \times S, \mathcal{O}^+) \simeq \operatorname{Cond}_p(D_X)(S)$$

on extremally disconnected parameters, and the corresponding derived evaluation for every profinite S . The comparisons retain the finite-level unit $\mathbb{Z}/p^r \rightarrow F_r$ and all formal coefficient maps.

For every profinite S , the resulting parameter comparison, in the form of [3, Lemma 3.6.1 and (3.6.2)], is

$$R\Gamma_{\text{cond}}((U_C \times S)_{\text{proét}}, \hat{\mathcal{O}}^+) \simeq R\text{Hom}_{\text{Solid}}(\mathbb{Z}[S]^\square, \mathcal{C}_{\text{proét}}(\mathcal{U})). \quad (\text{F.1})$$

Here $\text{Hom}_{\text{Solid}}$ is the internal Hom in solid modules, so both sides are solid complexes. The geometric action nerve is formed on the modern site. For a continuous semilinear group P acting on the charts, apply the modern coefficient functor to every term $X^\diamond \times P^q \times S$ and all maps between them. Transport this whole diagram along the preceding equivalences to the solid rigid model. This defines its continuous action, including composition coherences, and preserves its coefficient unit. In particular, take $P = \Gamma_K$ on U_C , and include the stabilizer and arithmetic actions in the global diagrams.

All ensuing constructions are made in the ringed topos of continuous equivariant objects on this site. When a cover or an exhaustion is present, its entire diagram of charts and restriction maps is included before taking resolutions; one may use the corresponding ringed diagram topos. Thus profinite parameters, action maps, and restriction maps belong to the objects on which the derived functors act. Restriction to a chart or parameter is the inverse-image pullback to the corresponding slice topos, with its inverse-image structure ring. The flat pullback comparison [9, Lemma 6.14] identifies the local $L\eta$ constructions for these slice restrictions, whose structure ring is the inverse image of $\mathcal{A}$. Transitions between formal models are retained as maps in the ringed diagram. Equation (F.1) identifies their derived sections with the continuous solid objects used in the main proof.

Choose a primitive p th root of unity in C . The invertible ideal

$$\mathcal{I} = (\zeta_p - 1)\mathcal{A}$$

is stable under the semilinear action. For $p = 2$, one has $\zeta_2 = -1$ and $\mathcal{I} = 2\mathcal{A}$. The relation $(\zeta_p - 1)^{p-1}\mathcal{A} = p\mathcal{A}$ then makes $\mathcal{H}^0(\mathcal{C}_{\text{proét}})$ $\mathcal{I}$ -torsion-free. For an $\mathcal{I}$ -torsion-free complex D , the décalage functor of [9, §6.1] is represented by

$$(\eta_{\mathcal{I}}D)^j = \ker(D^j \rightarrow D^{j+1}/\mathcal{I}D^{j+1}) \otimes_{\mathcal{A}} \mathcal{I}^{\otimes j}.$$

This expression retains the ideal line and is natural on the whole diagram.

F.2 Logarithmic Hodge cohomology in low degrees

Write $\Omega_{\mathcal{U}/\mathcal{O}_K, \log}^1$ for the p -adic completion of the relative logarithmic differentials, with the canonical log structures defined by invertibility on the generic fibers. Its exterior powers are denoted $\Omega_{\mathcal{U}/\mathcal{O}_K, \log}^j$. These are continuous formal logarithmic differentials; on an affine chart, $\Omega_{A/\mathcal{O}_K, \log}^j$ denotes their module of sections.

The underlying integral logarithmic Hodge comparison is due to Bhatt, Morrow, and Scholze [9, Theorem 8.3] and Česnavičius and Koshikawa [10, Theorem 4.11]. Its ordinary Galois-equivariant form is recorded in [3, Theorem 5.2.5 and Corollary 5.2.6]. The proof below supplies its continuous enhancement on the original formal site and the parameterized action diagram.

Lemma F.2. *For $0 \leq j \leq 3$, there are canonical isomorphisms of sheaves in the continuous semilinear equivariant category*

$$\mathcal{H}^j(L\eta_{\mathcal{I}}\mathcal{C}_{\text{proét}}) \simeq \underline{\Omega_{\mathcal{U}/\mathcal{O}_K, \log}^j \hat{\otimes}_{\mathcal{O}_K} \mathcal{O}_C \{-j\}}.$$

The right side has its p -adic condensed topology. Here $\mathcal{O}_C\{1\} = T_p(\Omega_{\mathcal{O}_C/\mathbb{Z}_p}^1)$, and $\{-j\}$ denotes the j th tensor power of its dual line. The isomorphism in degree zero identifies the coefficient unit.

Proof. The parameter construction above proves the affinoid condensation comparison used in [3, Lemma 3.7.1]:

$$\mathcal{C}_{\text{proét}}(\mathfrak{U}) \simeq \text{Cond}_p(R\Gamma((U_C)_{\text{proét}}^{\text{old}}, \widehat{\mathcal{O}}^+)). \quad (\text{F.2})$$

Its rigid-site proof uses finite clopen parameters and derived p -completion. The comparison with the modern coefficient model supplies restrictions, units, and the continuous action nerve.

The functor Cond_p is exact. If all cohomology groups of D are killed by p^a for one a , then γ^*D is already derived p -complete. Indeed, the inverse system with transitions multiplication by p has pro-zero cohomology. Exact products of solid modules and the Milnor sequence give zero derived limit. The derived-completeness criterion therefore gives $\text{Cond}_p(D) = \gamma^*D$. This argument also applies to an unbounded complex.

On a sufficiently small semistable chart, the ordinary toric edge map of [10, (3.3.1) and Lemma 3.7] is

$$D_{\text{tor}} := R\Gamma_{\text{cts}}(\mathbb{Z}_p^d, R_\infty) \longrightarrow R\Gamma((U_C)_{\text{proét}}^{\text{old}}, \widehat{\mathcal{O}}^+).$$

The source is the length- d toric Koszul complex, and its cone has cohomology annihilated by $\mathfrak{m}_C$. After applying Cond_p , that cone is its discrete condensation. A profinite family in $\mathfrak{m}_C$ acts on it through the discrete quotient $\mathfrak{m}_C/p\mathcal{O}_C$ and has finite image. Each value kills the ordinary cohomology. Thus the condensed edge map is internally an almost isomorphism.

Put $f = \zeta_p - 1$ locally. The terms of D_{tor} are finite sums of the complete torsion-free ring R_∞ . For

$$D'_j = \ker(D_{\text{tor}}^j \longrightarrow D_{\text{tor}}^{j+1}/f)$$

one has $fD_{\text{tor}}^j \subset D'_j \subset D_{\text{tor}}^j$ and

$$p^{r+1}D_{\text{tor}}^j \subset p^r D'_j \subset p^r D_{\text{tor}}^j.$$

Since $p \in (f)$, the quotient D_{tor}^{j+1}/f is discrete. The differential into this quotient is continuous, so D'_j is a closed submodule of the complete module D_{tor}^j . The displayed inclusions show that its induced and intrinsic p -adic topologies agree. Consequently D'_j is complete and torsion-free. Condensation preserves the defining kernel. The bounded complexes consequently satisfy

$$\text{Cond}_p(\eta_f D_{\text{tor}}) \simeq \eta_f \text{Cond}_p(D_{\text{tor}}).$$

The decomposition in [10, (3.2.2) and Proposition 3.8] gives

$$\mathcal{H}^j(\text{Cond}_p(D_{\text{tor}})) \simeq \underline{A}_C^{\oplus \binom{d}{j}} \oplus \gamma^* T_j, \quad fT_j = 0,$$

Here A_C is the base-changed chart ring. The modules T_j and A_C/fA_C have no nonzero $\mathfrak{m}_C$ -torsion. The displayed cohomology object and its quotient by f therefore have no nonzero internal almost-zero subobject.

Suppose $M \rightarrow N$ is an almost isomorphism with this property for M and M/fM . Its kernel is zero. For a local section y of N , the element fy lies in M . Its class in M/fM is almost zero, hence zero, so $fy = fx$ for a local x in M . It follows that $M/M[f] \simeq N/N[f]$. Apply this to the cohomology of the condensed edge map. The formula of [9, Lemma 6.4] now gives

$$L\eta_{\mathcal{I}} \text{Cond}_p(D_{\text{tor}}) \simeq L\eta_{\mathcal{I}} \mathcal{C}_{\text{proét}}(\mathfrak{U}).$$

The logarithmic calculation [10, Proposition 4.4, Remark 4.5, and Theorem 4.11] is applied to $\mathfrak{V}_{\mathcal{O}_C}$ for each affine étale chart $\mathfrak{V} \rightarrow \mathfrak{U}$. It identifies the affine cohomology presheaves with the claimed continuous logarithmic forms. By Remark 4.5 of loc. cit., these presheaves are

already sheaves; their chart modules are finite projective and p -complete. The calculation respects restriction and gives the displayed condensed topology. Take the torsion-free resolution of [9, Lemma 6.1] in the ringed presheaf topos with its parameters. Exact sheafification preserves the finite kernel formula for $\eta_{\mathcal{I}}$, so these chartwise calculations compute the stated cohomology sheaves on the original formal étale site.

The action follows intrinsically from

$$\mathcal{H}^j(L\eta_{\mathcal{I}}\mathcal{C}_{\text{proét}}) \simeq (\mathcal{H}^j(\mathcal{C}_{\text{proét}})/\mathcal{H}^j(\mathcal{C}_{\text{proét}})[\mathcal{I}]) \otimes \mathcal{I}^{\otimes j}. \quad (\text{F.3})$$

In degree one, the Kummer action contributes $\chi(\sigma)^{-1}$ and the ideal line contributes $\sigma(f)/f$. Their product is the action on $\mathcal{O}_C\{-1\}$: by [9, Definition 8.2], for $\omega = (d \log \zeta_p)_r$, the generator ω/f of $\mathcal{O}_C\{1\}$ transforms by the inverse factor. Exterior powers give the action in degree j . The cotangent-class and cup-product construction of [9, Propositions 8.15 and 8.17], with its logarithmic extension in [10, Theorem 4.11], respects semilinear isomorphisms preserving the intrinsic log structure $\mathcal{O} \cap (\mathcal{O}[1/p])^\times$ and $\mathcal{I}$.

On the finite projective chart modules, the comparison and its inverse are linear and continuous. For a profinite family of automorphisms, its two equivariance maps are continuous maps into Hausdorff modules and agree pointwise, so they agree as condensed maps. Their derived coherences are supplied by the geometric action nerve described above. $\square$

F.3 A truncated cofiber bound

The canonical décalage map is constructed in [9, Lemmas 6.9–6.10] and recalled in [3, Lemma 5.5.1]. We prove the following truncated-cofiber bound in the equivariant diagram category for the low-degree comparison.

Lemma F.3. *Let $\mathcal{A}$ be a ring in a ringed topos, possibly with continuous semilinear action and a diagram of restrictions. Let $\mathcal{I} \subset \mathcal{A}$ be a stable invertible ideal. Suppose $\mathcal{C} \in D^{\geq 0}(\mathcal{A})$ and $\mathcal{H}^0(\mathcal{C})$ is $\mathcal{I}$ -torsion-free. For the canonical map $a : L\eta_{\mathcal{I}}\mathcal{C} \rightarrow \mathcal{C}$, put $E = \text{cofib}(a)$. Then $E \in D^{\geq 1}$, and for each integer $m \geq 0$ the ideal-multiplication map*

$$\mathcal{I}^{\otimes m} \otimes_{\mathcal{A}}^L \tau_{\leq m} E \longrightarrow \tau_{\leq m} E$$

is zero in the equivariant diagram derived category. For $\mathcal{I} = (\zeta_p - 1)\mathcal{A}$, multiplication by p^{e_p} on $\tau_{\leq 3} E$ is zero in the same equivariant diagram derived category.

Proof. Take the strongly K -flat resolution of [9, Lemma 6.1] in the whole equivariant ringed diagram topos. Its terms are $\mathcal{I}$ -torsion-free. Replace it by its lower good truncation in degree zero. The new zero term is still $\mathcal{I}$ -torsion-free: it is an extension of $\mathcal{H}^0(\mathcal{C})$ by $\text{im } d^0$, which is a submodule of a torsion-free term. Call this nonnegative representative $P^\bullet$. Although truncation need not preserve flatness, [9, Lemma 6.4 and Corollary 6.5] allow any termwise $\mathcal{I}$ -torsion-free representative to compute $L\eta_{\mathcal{I}}$.

The comparison of [9, Lemmas 6.9–6.10] is now the inclusion of complexes $\eta_{\mathcal{I}}P^\bullet \hookrightarrow P^\bullet$. Hence the quotient complex $Q = P^\bullet/\eta_{\mathcal{I}}P^\bullet$ represents E . For every $j \geq 0$, the definition gives

$$\mathcal{I}^{j+1}P^j \subset (\eta_{\mathcal{I}}P^\bullet)^j, \quad \mathcal{I}^{j+1}Q^j = 0.$$

For $j < m$, this already shows that $\mathcal{I}^m Q^j = 0$. At the top of the upper good truncation, let $\bar{x} \in \ker(Q^m \rightarrow Q^{m+1})$ and lift it locally to $x \in P^m$. The cycle condition gives $dx \in (\eta_{\mathcal{I}}P^\bullet)^{m+1} \subseteq \mathcal{I}^{m+1}P^{m+1}$. For $b \in \mathcal{I}^m$,

$$bx \in \mathcal{I}^m P^m, \quad d(bx) = b dx \in \mathcal{I}^{2m+1}P^{m+1} \subseteq \mathcal{I}^{m+1}P^{m+1}.$$

Thus $bx \in (\eta_{\mathcal{I}} P^\bullet)^m$, so $b\bar{x} = 0$. Every term of $\tau_{\leq m} Q$ is therefore annihilated by $\mathcal{I}^m$. Because the ideal is invertible, ordinary tensoring with it computes derived tensoring. The ideal-multiplication map is thus the zero map of actual complexes on the entire diagram.

In degree zero, a cycle in Q represented by $x \in P^0$ satisfies $dx \in (\eta_{\mathcal{I}} P^\bullet)^1 \subset \mathcal{I}P^1$. Thus $x \in (\eta_{\mathcal{I}} P^\bullet)^0$. There are no negative terms, so $E \geq 1$. Equivalently, a is an isomorphism on $\mathcal{H}^0$ and an injection on $\mathcal{H}^1$. If $\mathcal{I} = (f)$ locally, the latter is $f : \mathcal{H}^1(\mathcal{C})/\mathcal{H}^1(\mathcal{C})[f] \rightarrow \mathcal{H}^1(\mathcal{C})$.

Finally $\mathcal{I}^{p-1} = p\mathcal{A}$ and $e_p(p-1) \geq 3$. Hence the equivariant inclusion

$$p^{e_p} \mathcal{A} = \mathcal{I}^{e_p(p-1)} \subseteq \mathcal{I}^3$$

factors multiplication by p^{e_p} through the zero map with $m = 3$. This factorization is formed on the entire continuous equivariant restriction diagram. $\square$

The exponent m is sharp in general. For $m \geq 1$, take $\mathcal{A} = \mathbb{Z}_p$, $\mathcal{I} = (p)$, and $\mathcal{C} = \mathbb{Z}_p[-m]$. Then $E = (\mathbb{Z}_p/p^m)[-m]$, on which p^{m-1} is nonzero.

If an exact functor F preserves $D^{\geq 0}$, then

$$\tau_{\leq 3} F(E) \simeq \tau_{\leq 3} F(\tau_{\leq 3} E).$$

Derived sections, continuous invariants, and derived inverse limits have this property. The preceding proof applies to a cover or exhaustion before its limit is taken, so its strict zero map retains the restriction coherences and passes through that limit. Multiplication by p^{e_p} is therefore zero on $\tau_{\leq 3} F(E)$ after each of these functors, with the continuous actions and restriction coherences retained.

The object truncated here is the cofiber E . Truncating $\mathcal{C}$ first can lose the contribution of $\ker \mathcal{H}^4(a)$ to $\mathcal{H}^3(E)$. For instance, for $A = \mathbb{F}_p[[f]]$ and $\mathcal{C} = (A/f^2)[-4]$, one has $\mathcal{H}^4(L\eta_f \mathcal{C}) = A/f$ and $\mathcal{H}^4(a) = 0$, so $\mathcal{H}^3(E) = A/f$, whereas $\tau_{\leq 3} \mathcal{C} = 0$.

F.4 The affine comparison and tame uniformity

Proof of Lemma F.1. The formal-to-generic-fiber unit factors naturally as

$$\mathcal{A} \longrightarrow L\eta_{\mathcal{I}} \mathcal{C}_{\text{proét}} \xrightarrow{a} \mathcal{C}_{\text{proét}}.$$

Indeed $L\eta_{\mathcal{I}} \mathcal{A} = \mathcal{A}$, and naturality of the canonical map a identifies its composite with the original coefficient unit. Apply derived formal sections and Γ_K -invariants. The Galois torsor $U_C \rightarrow U_K$ has Čech nerve $U_C \times \Gamma_K^q$. Pro-étale descent and (F.1) give

$$\begin{aligned} R\Gamma_{\text{cond}}((U_K)_{\text{proét}}, \widehat{\mathcal{O}}^+) &\simeq \text{Tot}_q R\Gamma_{\text{cond}}((U_C \times \Gamma_K^q)_{\text{proét}}, \widehat{\mathcal{O}}^+) \\ &\simeq R\Gamma_{\text{cond}}((U_C)_{\text{proét}}, \widehat{\mathcal{O}}^+)^{h\Gamma_K}. \end{aligned}$$

Naturality of the parameter comparison identifies the face and degeneracy maps with the continuous Galois cochains, as in the proof of [3, Proposition 3.6.3]. Thus the resulting rightmost object is the pro-étale cohomology of U_K .

Lemma E.2, applied to A , identifies the leftmost object with $A \widehat{\otimes}_{\mathcal{O}_K} R\Gamma_{\text{cts}}(\Gamma_K, \mathcal{O}_C)$. The cofiber of the field map $\alpha_{0,K}$ lies in $D^{\geq 1}$ and has bounded p -torsion cohomology [3, Theorem 4.3.8 and (5.3.2)]. Flatness of A preserves these bounds. The discrete-condensation and completion construction above preserves the bounds on this uniformly torsion cofiber and supplies their solid form.

The cofiber of $\mathcal{A} \rightarrow L\eta_{\mathcal{I}} \mathcal{C}_{\text{proét}}$ lies in $D^{\geq 1}$. Its low Postnikov pieces are the positive logarithmic forms in Lemma F.2. On an affine chart, these finite projective modules have no

higher coherent cohomology. For each $1 \leq j \leq 3$, the projection formula therefore identifies the corresponding Galois piece with

$$\underline{\Omega}_{A/\mathcal{O}_K, \log}^j \hat{\otimes}_{\mathcal{O}_K} R\Gamma_{\text{cts}}(\Gamma_K, \mathcal{O}_C\{-j\})[-j].$$

For the positive integers $1 \leq j \leq 3$, the estimates in [3, Theorem 4.4.3 and the proof of Lemma 5.4.1] give $H_{\text{cts}}^0(\Gamma_K, \mathcal{O}_C\{-j\}) = 0$ and a uniform p -power bound for its positive cohomology, depending only on the tame class of K and on j . Here $j = 0$ is the field piece already treated above, with $H_{\text{cts}}^0(\Gamma_K, \mathcal{O}_C) = \mathcal{O}_K$. The finite Postnikov filtration through degree three therefore gives the required torsion bound for the positive Hodge pieces.

For the remaining map a , Lemma F.3 makes multiplication by p^{e_p} zero on its truncated cofiber. It remains zero after derived formal sections and Galois invariants, in the continuous equivariant diagram category. All three comparison cofibers lie in $D^{\geq 1}$. The cofiber triangles for their composite retain that lower bound, and their long exact sequences add the torsion exponents in each of the three degrees. The resulting composite is exactly $\alpha_{\mathbb{U}}$, since its maps were formed by pullback and multiplication with the chosen field map. This proves the cofiber assertion, including the degree-one injectivity needed for the lower bound.

We spell out the indexing in the tame-uniformity assertion. Let K_{∞}/K be the chosen ramified $\mathbb{Z}_p$ -tower, and choose N such that its tail over K_N is sufficiently ramified. For a finite tame Galois extension L/K , write

$$L \cap K_{\infty} = K_s.$$

This intersection is unramified, since its degree is a power of p and it is tame. The degree- p^j layers over L are therefore $L_j = LK_{s+j}$. Initial jumps equal to -1 stay equal to -1 ; the positive jumps scale by $e_{L/K}$, by [3, Proposition 4.1.5]. Since the chosen tail is totally ramified, $s \leq N$.

The sufficiently ramified tail over L can be taken at $N' = N - s$. To check this directly, put $e = e_{LK_N/K_N}$. At each degree- p step in the totally ramified tail, tame base change gives

$$d'_j = e d_j - (e - 1)(p - 1),$$

where the relative different exponents are measured by the normalized integer-valued valuations of the upper fields. Consequently

$$d_j \geq e_{K_N}(p^j - 1) + (p - 1) \implies d'_j \geq e_{LK_N}(p^j - 1) + (p - 1).$$

Thus $N' \leq N$, as required for the field bound of [3, Theorem 4.3.8].

This also bounds the actual pullback of the chosen degree-one class. Restriction from the original tower to its index- p^s subgroup multiplies a normalized degree-one generator by p^s . Its additional cofiber is killed by p^s , while the tail threshold has dropped to $N - s$. The combined exponent can therefore be bounded using the original N , uniformly in L . For twists, apply the argument separately to the full cyclotomic extension T/K . Put

$$U_{\text{pr}} = \begin{cases} 1 + p\mathbb{Z}_p, & p \neq 2, \\ 1 + 4\mathbb{Z}_2, & p = 2, \end{cases}$$

and let K_{pr} be the fixed field of $\chi_{\text{cycl}}^{-1}(U_{\text{pr}})$. As in [3, §4.4], write $\chi_{\text{cycl}}(\text{Gal}(T/K_{\text{pr}})) = 1 + p^r\mathbb{Z}_p$. Write T_j/K_{pr} for the degree- p^j layers, and choose N_{pr} such that $T/T_{N_{\text{pr}}}$ is sufficiently ramified. For the tame base change set $L_{\text{pr}} = LK_{\text{pr}}$ and $L_{\text{pr}} \cap T = T_{s_{\text{pr}}}$. The extension $L_{\text{pr}}/K_{\text{pr}}$ is tame, so $T_{s_{\text{pr}}}/K_{\text{pr}}$ is unramified and $s_{\text{pr}} \leq N_{\text{pr}}$. The principal-unit tower over L_{pr} consequently has character depth $r + s_{\text{pr}}$ and sufficiently ramified threshold $N_{\text{pr}} - s_{\text{pr}}$. The sum $N_{\text{pr}} + r$

is unchanged. The finite quotient $\text{Gal}(L_{\text{pr}}/L)$ embeds in $\text{Gal}(K_{\text{pr}}/K)$: its order is prime to p for odd p , and is at most two for $p = 2$. Thus averaging, or the finite low-degree Hochschild–Serre filtration when $p = 2$, introduces only a uniform additional p -power bound. The proof of [3, Theorem 4.4.3] therefore gives uniform positive-twist bounds in the degrees used here. The index s_{pr} belongs to this principal-unit tower; it need not equal the index s of the maximal $\mathbb{Z}_p$ -quotient tower above. We allow enlarged constants and require no optimal numerical value for them. The décalage exponent e_p is independent of L . After a tame Galois closure, semilinear invariants are exact by [3, Lemma 5.6.1]. Concretely, choose $c \in \mathcal{O}_L$ with $\text{Tr}_{L/K}(c) = 1$. The operator $x \mapsto \sum_g g(cx)$ gives invariant lifts of invariant sections in a quotient. The trace-one element exists also when the unramified degree is divisible by p : the residue-field trace is surjective, and in the totally tamely ramified part the ramification degree is a unit. The operator acts continuously on the same parameter spaces and introduces no factor depending on $[L : K]$. This proves the asserted tame uniformity and the low-degree form of [3, Lemma 5.6.2] used in the closed-ball exhaustion. $\square$

For the two global comparisons, choose their base-field degree-one maps separately. On Lubin–Tate space, one can obtain the chosen map by pullback from $\mathbb{Q}_p$; the unramified coefficient field has the same cyclotomic $\mathbb{Z}_p$ -quotient. Every arithmetic stabilizer automorphism is a $\mathbb{Q}_p$ -automorphism, so this generator is fixed, including under arithmetic Frobenius. On Drinfeld space, choose its base-field map over $\mathbb{Q}_p$ and use it on every formal chart. Its generator is fixed by $\text{GL}_n(\mathbb{Z}_p)$. Pullback, coefficient units, and multiplication commute with the face and degeneracy maps of their action groupoids. They also commute with the restriction maps within each cover or exhaustion. These are the continuous equivariant maps appearing in Lemma E.1.

G Witt cohomology and the source weight at height three

The height-three case of Lemma 2.7 uses the Witt and ASW source-norm calculation of Lemma G.1 below.

Throughout this appendix, $n = 3$ and p is arbitrary. Put

$$S_3 = \ker(\text{Nrd}_3 : \mathcal{O}_{D_3}^\times \longrightarrow \mathbb{Z}_p^\times).$$

The source norm quotient acts on $R\Gamma_{\text{cts}}(S_3, W_m(\widehat{L}))$ and on $R\Gamma_{\text{cts}}(S_3, \text{ASW}(\widehat{L}))$, with the target and arithmetic components held fixed. These are the actual actions induced by $G_{3,2}$. All cohomology in this appendix has its continuous solid enhancement; $\mathcal{H}^j$ denotes its cohomology object in the category of solid modules. For a diamond over a perfectoid field, this is the enhancement of [2, Remarks 3.2.3 and 3.2.5]; finite Witt coefficients are formed on the perfectoid coefficient rings before taking derived sections.

The residue-field cofiber calculation is [2, Theorem 3.6.10]. The following lemma extends it to Witt and ASW coefficients and computes the integral source-norm action at height three. The Witt-coefficient extension is suggested in [2, Remark 3.6.12(2)]; here we give the construction at height three.

Lemma G.1. *For every $m \geq 1$, the constant unit identifies degree zero of $R\Gamma_{\text{cts}}(S_3, W_m(\widehat{L}))$ with $W_m(\overline{\mathbb{F}}_p)$. Its only other cohomology is*

$$T_m = \mathcal{H}^3 R\Gamma_{\text{cts}}(S_3, W_m(\widehat{L})),$$

a free rank-one solid $W_m(\overline{\mathbb{F}}_p)$ -module. The residual source norm $a \in \mathbb{Z}_p^\times$ acts on T_m by the scalar $a^{-2} \in (\mathbb{Z}/p^m)^\times$.

The cofiber of the specified ASW unit is concentrated in degree three:

$$\mathrm{cofib}\left(\mathbb{Z}_p \longrightarrow R\Gamma_{\mathrm{cts}}(S_3, \mathrm{ASW}(\widehat{L}))\right) \simeq T_{\mathrm{ASW}}[-3],$$

where T_{ASW} is a free rank-one solid $\mathbb{Z}_p$ -module and source norm a acts by a^{-2} . Norm -1 acts trivially. For $p = 2$, norm 5 acts by 25^{-1} ; for odd p , norm $1 + p$ acts by $(1 + p)^{-2}$.

Proposition G.3 supplies the relative cup-product duality on the Drinfeld half-plane, and Lemma G.4 computes translation cohomology with its full unit action. Frobenius descent then transports the resulting weight to the actual source quotient.

G.1 A relative Witt duality on the half-plane

Let $F/\mathbb{Q}_p$ be the unramified quadratic extension and let C be an algebraically closed perfectoid field containing F . View $\mathbf{P}_C^1$ and $\mathbb{H}_{F,C}^{1,\diamond}$ as diamonds. We follow the locally constant function notation of [2, Theorem 3.4.1] and the solid-linear distribution convention of [2, Definition 3.4.5]. The definitions below give their analogues over a general solid coefficient ring, including the finite Witt rings $W_m(C^\flat)$. For a profinite set P and a solid coefficient ring R , write

$$\mathrm{LC}(P, R) = \mathrm{colim}_{\mathcal{P}} \prod_{V \in \mathcal{P}} R,$$

where $\mathcal{P}$ runs over finite clopen partitions of P , ordered by refinement, and the colimit is taken in solid R -modules. A tuple $(r_V)_{V \in \mathcal{P}}$ assigns the constant value r_V to each piece V ; refinement repeats this value on its subpieces. Local constancy concerns the P -variable, while the solid structure of the coefficients is retained. For topological coefficients, this construction differs from all continuous functions in general; see [2, Remark 3.5.2]. For the locally profinite space F , define $\mathrm{LC}_c(F, R)$ as the colimit of $\mathrm{LC}(K, R)$ over compact open subsets $K \subset F$, with extension by zero. For $R = C^\flat$, this is the compactly supported model used in [2, proof of Proposition 3.5.6]. Define

$$\mathrm{Dist}(F, R) = \underline{\mathrm{Hom}}_R(\mathrm{LC}_c(F, R), R).$$

The Hom here is the solid internal linear Hom . The left action on distributions is pushforward: $(g\mu)(f) = \mu(f \circ g)$. We use the corresponding inverse-pullback left action on cohomology. The notation (-1) over C denotes the ordinary arithmetic Tate twist. Thus $W_m(C^\flat)(-1)$ is $W_m(C^\flat) \otimes_{\mathbb{Z}/p^m} \mathbb{Z}/p^m(-1)$.

The following Witt-completion argument starts from affinoid perfectoid acyclicity [13, Proposition 8.8].

Lemma G.2. *For a diamond Z over C , the complex $R\Gamma(Z, W\mathcal{O}^\flat)$ is derived p -complete, and the coefficient quotient induces natural equivalences*

$$R\Gamma(Z, W\mathcal{O}^\flat) \otimes_{W(C^\flat)}^{\mathbf{L}} W_m(C^\flat) \simeq R\Gamma(Z, W_m\mathcal{O}^\flat).$$

These statements hold with profinite parameters and commute with pullback and continuous actions.

Proof. On an affinoid perfectoid test object Y over C , write $R_Y = \mathcal{O}^\flat(Y)$. Tilting and [13, Proposition 8.8] give $R\Gamma_v(Y, \mathcal{O}^\flat) = R_Y[0]$. This is vanishing for $\mathcal{O}^\flat$, rather than almost vanishing for its integral subring. Induction through the additive Verschiebung exact sequences gives

$$R\Gamma_v(Y, W_m\mathcal{O}^\flat) = W_m(R_Y)[0].$$

Define $\mathcal{Q} = R\varprojlim_m W_m \mathcal{O}^b$ in the derived category of sheaves. Derived sections preserve limits. The Witt truncation maps are surjective, so

$$R\Gamma_v(Y, \mathcal{Q}) = R\varprojlim_m W_m(R_Y) = W(R_Y)[0].$$

Affinoid perfectoid test objects form a basis for the v -site; this identifies the natural map $W\mathcal{O}^b \rightarrow \mathcal{Q}$ with an equivalence. Since R_Y is perfect, $W(R_Y)$ is p -torsion-free and $W(R_Y)/p^m = W_m(R_Y)$. Hence the cofiber of multiplication by p^m on $W\mathcal{O}^b$ is $W_m \mathcal{O}^b$. Exactness of derived sections identifies the same cofiber after $R\Gamma(Z, -)$, proving the displayed base-change equivalence and derived completeness.

For an extremally disconnected profinite parameter U , the product $Y \times U$ is again affinoid perfectoid, and the same calculation applies to its perfect coefficient ring. It therefore proves the assertions in the solid enhancement. All maps used are Witt quotient, truncation, and pullback maps, so they commute with the full action diagrams. $\square$

The length-one pairing is the projective-line and half-plane duality of [2, Theorems 3.4.1 and 3.4.6]. The next proposition constructs the compatible finite-Witt pairings and their p -complete limit.

Proposition G.3. *The solid cohomology of the Drinfeld half-plane with finite Witt coefficients is*

$$\mathcal{H}^j R\Gamma(\mathbb{H}_{F,C}^{1,\diamond}, W_m \mathcal{O}^b) \simeq \begin{cases} W_m(C^b), & j = 0, \\ \text{Dist}(F, W_m(C^b))(-1), & j = 1, \\ 0, & j \neq 0, 1. \end{cases}$$

The degree-zero identification is the constant unit. The degree-one identification is equivariant for the Borel group fixing infinity; in particular it respects translations and dilations of F . These identifications commute with length reduction.

Proof. We construct a relative complex for the boundary $\mathbf{P}^1(F)$ and a pairing with $R\Gamma(\mathbb{H}_{F,C}^{1,\diamond}, W\mathcal{O}^b)$. Its reduction modulo p is the duality in [2, Theorems 3.4.1 and 3.4.6].

The projective-line trace. The finite constant coefficient units and their derived inverse limit give a morphism

$$W(C^b) \otimes_{\mathbb{Z}_p}^{\mathbf{L}} R\Gamma_{\text{ét}}(\mathbf{P}_C^1, \mathbb{Z}_p) \longrightarrow R\Gamma(\mathbf{P}_C^1, W\mathcal{O}^b).$$

Here $\mathbb{Z}_p$ -cohomology on the left is the derived inverse limit of finite constant cohomology. It is a perfect complex, with terms $\mathbb{Z}_p$ in degree zero and $\mathbb{Z}_p(-1)$ in degree two. The left complex is derived p -complete because its constant cohomology complex is perfect and $W(C^b)$ is p -complete. Lemma G.2 gives derived completeness and the finite Witt quotients on the right. Reduction of the displayed morphism modulo p is the primitive comparison used in the proof of [2, Theorem 3.4.1]. Derived Nakayama therefore makes it an equivalence. The usual projective-line trace gives compatible maps

$$R\Gamma(\mathbf{P}_C^1, W_m \mathcal{O}^b) \longrightarrow W_m(C^b)(-1)[-2], \tag{G.1}$$

and cohomology of the source is $W_m(C^b)$ in degree zero and $W_m(C^b)(-1)$ in degree two. Geometric automorphisms of $\mathbf{P}_C^1$ preserve this trace. The Tate factor records its arithmetic action; translations and dilations over F act trivially on that factor.

The boundary complex. Use the cofiltered system of open neighborhoods U_i of $\mathbf{P}^1(F)$ in the proof of [2, Theorem 3.4.1]. A cofinal shrinking sequence consists of disjoint finite unions of discs. Positive-degree $\mathcal{O}^b$ -cohomology of a disc becomes zero after a sufficiently small restriction, by [2, Lemma 3.3.6], and its degree-zero cohomology is C^b . The additive exact sequences

$$0 \longrightarrow W_{m-1}\mathcal{O}^b \xrightarrow{V} W_m\mathcal{O}^b \longrightarrow \mathcal{O}^b \longrightarrow 0$$

extend this assertion to every fixed Witt length. More explicitly, if a map is zero on the cohomology of each associated graded object of a finite filtration, a composite of as many such maps as there are filtration steps is zero. Thus additional shrinkings kill all positive-degree cohomology at length m . Degree zero on each disc is $W_m(C^b)$, because Witt vectors commute with limits of rings. Consequently

$$\operatorname{colim}_i R\Gamma(U_i, W_m\mathcal{O}^b) \simeq \operatorname{LC}(\mathbf{P}^1(F), W_m(C^b)). \quad (\text{G.2})$$

The right side is in degree zero. We use the full neighborhood system for its group action, and the shrinking sequence only for this calculation. A compact family of translations or dilations admits common refinements of these neighborhoods, so the construction also respects profinite parameters of the continuous action.

Define the relative boundary complex

$$K_m = \operatorname{fib} \left(R\Gamma(\mathbf{P}_C^1, W_m\mathcal{O}^b) \longrightarrow \operatorname{colim}_i R\Gamma(U_i, W_m\mathcal{O}^b) \right).$$

The map in degree zero restricts constant functions. Hence

$$\mathcal{H}^1(K_m) = \operatorname{LC}(\mathbf{P}^1(F), W_m(C^b))/W_m(C^b), \quad \mathcal{H}^2(K_m) = W_m(C^b)(-1), \quad (\text{G.3})$$

and all its other cohomology vanishes. Subtracting the value at infinity identifies the first module with $\operatorname{LC}_c(F, W_m(C^b))$, equivariantly for the Borel group. The neighborhood colimit in (G.2) is the overconvergent boundary object of [2, Proposition 3.2.6 and Remark 3.2.7]; it retains locally constant functions.

The relative cup product. For each i , the open cover $\mathbb{H}_{F,C}^{1,\diamond} \cup U_i = \mathbf{P}_C^1$ identifies

$$\operatorname{fib}(R\Gamma(\mathbf{P}_C^1, W_m\mathcal{O}^b) \rightarrow R\Gamma(U_i, W_m\mathcal{O}^b)) \simeq \operatorname{fib}(R\Gamma(\mathbb{H}_{F,C}^{1,\diamond}, W_m\mathcal{O}^b) \rightarrow R\Gamma(\mathbb{H}_{F,C}^{1,\diamond} \cap U_i, W_m\mathcal{O}^b)).$$

The second relative complex is a module over $R\Gamma(\mathbb{H}_{F,C}^{1,\diamond}, W_m\mathcal{O}^b)$. Its multiplication, followed by the map to projective-line cohomology and (G.1), gives a pairing

$$R\Gamma(\mathbb{H}_{F,C}^{1,\diamond}, W_m\mathcal{O}^b) \otimes_{W_m(C^b)}^{\mathbf{L}} K_m \longrightarrow W_m(C^b)(-1)[-2]. \quad (\text{G.4})$$

Mayer–Vietoris, multiplication, and the coefficient unit provide these maps before passing to cohomology. They commute with length reduction, with restriction of neighborhoods, and with the action maps, faces, and degeneracies of translations and dilations.

Here is the compatibility of its modulo- p pairing with the one in [2, Theorems 3.4.1 and 3.4.6]. Work temporarily in that paper’s $\mathcal{O}^b$ six-functor category, and let $j : \mathbb{H}_{F,C}^{1,\diamond} \hookrightarrow \mathbf{P}_C^1$ and $j_i : U_i \hookrightarrow \mathbf{P}_C^1$. Set

$$C_i = \operatorname{fib}(\mathcal{O}_{\mathbf{P}_C^1}^b \rightarrow (j_i)_*\mathcal{O}_{U_i}^b).$$

The counit $j_{!}j^*C_i \rightarrow C_i$ is an equivalence: its restriction to $\mathbb{H}_{F,C}^{1,\diamond}$ is an equivalence, and both sides restrict to zero on U_i . These two opens cover $\mathbf{P}_C^1$. Under this identification, the

relative module multiplication is the cup product transported by $j_!$. Proposition 3.2.6 of loc. cit. identifies $\operatorname{colim}_i C_i$ with $j_! \mathcal{O}_{\mathbb{H}_{F,C}^{1,\diamond}}^b$. The map $f : \mathbf{P}_C^1 \rightarrow \operatorname{Spd} C$ is proper, so $f_* = f_!$ in this $\mathcal{O}^b$ six-functor category [2, Theorem 3.2.1(iv)(a)]. Since $f_!$ preserves colimits, applying f_* to this identification identifies K_1 with compactly supported cohomology of $\mathbb{H}_{F,C}^{1,\diamond}$. Thus the reduced relative pairing is the compact-support cup product. The primitive and Poincaré traces on the projective line are nonzero maps to $C^b(-1)[-2]$; their mapping space in degree zero is one-dimensional, so they differ by a unit. The reduced pairing therefore gives exactly the duality equivalence used in loc. cit., up to that harmless unit.

Complete lifting. Set $K = R\varprojlim_m K_m$. We retain the order of this inverse limit and the neighborhood colimit in the definition of K_m . The compact metrizable space $\mathbf{P}^1(F)$ has refining finite clopen partitions. Choose one child of each cell at every refinement; this splits the inclusions of their integer function modules. It gives a free basis for $\operatorname{LC}(\mathbf{P}^1(F), \mathbb{Z})$, including the constant function. Base change gives the same free clopen basis for every finite Witt coefficient ring. The inverse limit of these free modules is their p -adically completed free $W(C^b)$ -module, and its quotient by p^m is $\operatorname{LC}(\mathbf{P}^1(F), W_m(C^b))$. Together with the projective-line comparison this proves

$$K \otimes_{W(C^b)}^{\mathbf{L}} W_m(C^b) \simeq K_m.$$

Lemma G.2 also gives

$$R\Gamma(\mathbb{H}_{F,C}^{1,\diamond}, W\mathcal{O}^b) \otimes_{W(C^b)}^{\mathbf{L}} W_m(C^b) \simeq R\Gamma(\mathbb{H}_{F,C}^{1,\diamond}, W_m\mathcal{O}^b).$$

The compatible pairings (G.4) induce

$$R\Gamma(\mathbb{H}_{F,C}^{1,\diamond}, W\mathcal{O}^b) \longrightarrow R\operatorname{Hom}_{W(C^b)}(K, W(C^b)(-1)[-2]). \quad (\text{G.5})$$

Both sides are derived p -complete. Since $W(C^b)/p$ is the two-term cofiber of multiplication by p , it is a perfect $W(C^b)$ -module. Exactness of internal Hom therefore identifies the reduction of the right side with

$$R\operatorname{Hom}_{C^b}(K_1, C^b(-1)[-2]).$$

The modulo- p pairing just checked is an equivalence, so derived Nakayama proves (G.5) is an equivalence. The same perfect-base-change argument with $W(C^b)/p^m$ yields the finite-length duality. In particular this argument only uses the existing $\mathcal{O}^b$ duality at length one.

The two modules in (G.3) have internal dual concentrated in degree zero. For the first this follows from the free clopen basis; its internal dual is a product of copies of $W_m(C^b)$. Products are exact in solid modules, as can be checked on extremally disconnected projective parameter sets. Dualizing (G.3) with the trace target therefore gives $W_m(C^b)$ in degree zero and $\operatorname{Dist}(F, W_m(C^b))(-1)$ in degree one. The degree-zero map is the constant unit because the pairing was constructed from multiplication and the projective-line trace. Every comparison used was a specified coefficient or relative cup-product map, so Borel equivariance and length compatibility follow as well. $\square$

G.2 Continuous translation cohomology

The next calculation extends the translation-cohomology argument in [2, the proof of Proposition 3.5.6] to finite Witt coefficients, retaining the integral dilation action.

Lemma G.4. *For the additive locally profinite group F , there are equivalences*

$$\begin{aligned} R\Gamma_{\text{cts}}(F, W_m(C^\flat)) &\simeq W_m(C^\flat), \\ R\Gamma_{\text{cts}}(F, \text{Dist}(F, W_m(C^\flat))) &\simeq W_m(C^\flat)[-2]. \end{aligned}$$

The first coefficient action is trivial. For $a \in \mathbb{Z}_p^\times$, the dilation $x \mapsto ax$ acts on the degree-two line in the second equivalence by a^{-2} , with the pushforward distribution action and inverse-input action on group cochains. The equivalences commute with length reduction.

Proof. Give $W_m(C^\flat)$ its finite-Witt topology; in particular, $p^m W_m(C^\flat) = 0$. Choose a $\mathbb{Z}_p$ -basis of $\mathcal{O}_F$, with additive generators γ_1, γ_2 . Dualizing the finite clopen partitions of each coset gives

$$\text{Dist}(F, W_m(C^\flat)) \simeq \prod_{F/\mathcal{O}_F} W_m(C^\flat)[[\mathcal{O}_F]], \quad W_m(C^\flat)[[\mathcal{O}_F]] = \varprojlim_r W_m(C^\flat)[\mathcal{O}_F/p^r \mathcal{O}_F]. \quad (\text{G.6})$$

The finite group modules carry their coefficient topology, so this inverse limit and the product are taken in solid modules.

We compute continuous invariants using the augmentation resolution of $\Lambda = \mathbb{Z}_p[[\mathcal{O}_F]]$. Put $T_i = \gamma_i - 1$; then $\Lambda = \mathbb{Z}_p[[T_1, T_2]]$, and

$$0 \longrightarrow \Lambda \xrightarrow{\begin{pmatrix} -T_2 \\ T_1 \end{pmatrix}} \Lambda^{\oplus 2} \xrightarrow{(T_1, T_2)} \Lambda \longrightarrow \mathbb{Z}_p \longrightarrow 0$$

is exact in profinite modules. Its condensation is an exact resolution over the solid ring $\underline{\Lambda} = \mathbb{Z}_p[\mathcal{O}_F]^\square$ by [6, Lemma 2.13]. Modules over this solid ring are solid $\mathbb{Z}_p$ -modules with continuous $\mathcal{O}_F$ -action [6, Lemma 2.14]. Internal Hom from a rank-one free term is the underlying-module functor and is exact. Thus, for any such coefficient M , the resolution gives the continuous solid cochain model

$$R\Gamma_{\text{cts}}(\mathcal{O}_F, M) \simeq [M \xrightarrow{\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}} M^{\oplus 2} \xrightarrow{(-T_2, T_1)} M],$$

in degrees zero through two. This construction uses the profinite coefficient ring $\mathbb{Z}_p$; it applies to $W_m(C^\flat)$ with its given topology.

For $M = W_m(C^\flat)[[\mathcal{O}_F]]$, this complex is the power-series Koszul complex. To see the identification of completions, let $J = (T_1, T_2)$ in $W_m(C^\flat)[T_1, T_2]$ and put

$$I_r = ((1 + T_1)^{p^r} - 1, (1 + T_2)^{p^r} - 1).$$

For every N , the binomial coefficients in degrees $1, \dots, N-1$ are divisible by p^m when r is large, so $I_r \subseteq J^N$. Conversely, modulo p the two relations are $T_i^{p^r}$. Hence $J^{2p^r-1} \subseteq pW_m(C^\flat)[T_1, T_2] + I_r$, and $J^{m(2p^r-1)} \subseteq I_r$ because $p^m W_m(C^\flat) = 0$. These two systems of ideals are therefore cofinal, giving

$$W_m(C^\flat)[[\mathcal{O}_F]] \simeq W_m(C^\flat)[[T_1, T_2]].$$

This is also an identification of solid modules: every finite quotient on either side is a finite free $W_m(C^\flat)$ -module, and the comparison maps are continuous $W_m(C^\flat)$ -linear matrices. In particular, the power-series module retains the product topology of its coefficients.

Multiplication by T_1 and then by T_2 is a regular sequence on $W_m(C^\flat)[[T_1, T_2]]$, with quotient $W_m(C^\flat)$. Coefficient shifts and the corresponding projections are continuous, so the same exactness holds on profinite parameters. The displayed cochain model gives

$$R\Gamma_{\text{cts}}(\mathcal{O}_F, W_m(C^\flat)[[\mathcal{O}_F]]) \simeq W_m(C^\flat)[-2]. \quad (\text{G.7})$$

Continuous invariants preserve products. Thus (G.6) gives

$$R\Gamma_{\text{cts}}(\mathcal{O}_F, \text{Dist}(F, W_m(C^b))) \simeq \text{Map}(F/\mathcal{O}_F, W_m(C^b))[-2].$$

The quotient $F/\mathcal{O}_F$ acts by permutation of cosets. Changing a representative also introduces a lattice translation, which acts trivially on the augmentation quotient in degree two. The displayed module is consequently coinduced from the trivial subgroup of the discrete group $F/\mathcal{O}_F$. Its homogeneous bar complex contracts by inserting the identity coset. Taking its derived invariants proves $R\Gamma_{\text{cts}}(F, \text{Dist}(F, W_m(C^b))) \simeq W_m(C^b)[-2]$.

For trivial coefficients, the same Koszul resolution on each lattice $p^{-r}\mathcal{O}_F$ has zero differential and gives

$$\mathcal{H}^q R\Gamma_{\text{cts}}(p^{-r}\mathcal{O}_F, W_m(C^b)) \simeq W_m(C^b) \otimes_{\mathbb{Z}_p} \bigwedge^q (p^{-r}\mathcal{O}_F)^\vee, \quad 0 \leq q \leq 2.$$

Here the dual lattice is $\text{Hom}_{\mathbb{Z}_p}(p^{-r}\mathcal{O}_F, \mathbb{Z}_p)$. Restriction from $p^{-r-1}\mathcal{O}_F$ to $p^{-r}\mathcal{O}_F$ multiplies degree- q cohomology by p^q in the lattice-generator bases. Indeed, if γ generates the larger one-variable lattice, the resolution map sends a degree-one generator to $(1 + \gamma + \cdots + \gamma^{p-1})$ times the larger-lattice resolution generator; on trivial coefficients this is multiplication by p . The two-variable map is its tensor product.

Now $F = \bigcup_{r \geq 0} p^{-r}\mathcal{O}_F$, and continuous cochains on F are the inverse limit of the cochains on these open subgroups. The restriction maps in every cochain degree are surjective: a continuous function on the smaller compact open subgroup, with profinite parameters, extends by zero to the larger one. The ordinary inverse limit therefore also computes the derived inverse limit. In positive cohomological degrees the restriction systems are pro-zero, since $p^m W_m(C^b) = 0$; degree zero has identity transitions. It follows that $R\Gamma_{\text{cts}}(F, W_m(C^b)) \simeq W_m(C^b)$.

Finally the same augmentation resolution retains the actual dilation action. For $a \in \mathbb{Z}_p^\times$, pushforward of distributions induces

$$\varphi_a(T_i) = (1 + T_i)^a - 1 = T_i u_i(T_i), \quad u_i(0) = a.$$

On the chain resolution, dilation lifts as φ_a in degree zero, $u_i \varphi_a$ on the two degree-one generators, and $u_1 u_2 \varphi_a$ in degree two. Transport on internal Hom uses the inverse of this lift, as does the inverse-input action on group cochains. The resulting cochain map is therefore φ_a in degree zero, has components $u_i^{-1} \varphi_a$ in degree one, and is $(u_1 u_2)^{-1} \varphi_a$ in degree two. For example, $u_i^{-1} \varphi_a(T_i f) = T_i \varphi_a(f)$, and the analogous two-variable identity verifies the second differential. On the top augmentation quotient its scalar is a^{-2} . Dilation permutes $F/\mathcal{O}_F$, whose constant invariant section is fixed, so this scalar survives the quotient invariants. All constructions commute with Witt length reduction, proving the stated integral unit weight. $\square$

Proposition G.3 and the specified unit give the truncation triangle

$$W_m(C^b) \longrightarrow R\Gamma(\mathbb{H}_{F,C}^{1,\diamond}, W_m \mathcal{O}^b) \longrightarrow \text{Dist}(F, W_m(C^b))(-1)[-1].$$

Taking the additive F -invariants and applying Lemma G.4 shows that

$$\begin{aligned} \mathcal{H}^0 R\Gamma(\mathbb{H}_{F,C}^{1,\diamond}/F, W_m \mathcal{O}^b) &= W_m(C^b), \\ \mathcal{H}^3 R\Gamma(\mathbb{H}_{F,C}^{1,\diamond}/F, W_m \mathcal{O}^b) &= W_m(C^b)(-1), \end{aligned} \tag{G.8}$$

with no other cohomology. The dilation $z \mapsto az$ acts on the upper line by a^{-2} . The ordinary Tate factor in this formula has been retained throughout the translation calculation.

G.3 Frobenius descent and the actual source action

Proof of Lemma G.1. Choose a finite constant field k_0 containing $\mathbb{F}_{p^6}$, so both Honda groups and their endomorphisms are defined over k_0 , and put $q = |k_0|$. Use the punctured space $\mathrm{BC}_{k_0}(-1/2)^\times$ of Lemma D.11. The actual quotient presentation of Corollary D.9 identifies its pullback to $\overline{\mathbb{F}}_p$ with $[\mathrm{Spd} \widehat{L}/S_3]$.

For either k_0 or $\overline{\mathbb{F}}_p$, Lemma D.11 identifies residue cohomology in degrees zero and three with the constant field, with the specified degree-zero unit. The Witt exact sequences therefore give cohomology in these two degrees only at every finite length. The degree-zero term is W_m of the constant field. Write $T_m(k_0) = \mathcal{H}^3 R\Gamma(\mathrm{BC}_{k_0}(-1/2)^\times, W_m \mathcal{O}^b)$. Reduction $T_m(k_0) \rightarrow T_{m-1}(k_0)$ is surjective, and $V : T_{m-1}(k_0) \rightarrow T_m(k_0)$ is injective. Perfection makes Witt Frobenius invertible. The identity $p = V\varphi R$ consequently gives

$$pT_m(k_0) = VT_{m-1}(k_0), \quad T_m(k_0)/pT_m(k_0) \simeq k_0.$$

Its successive p -adic quotients each have rank one over k_0 . Lift a residue generator; nilpotent Nakayama makes it a generator over $W_m(k_0)$, and the filtration of length m makes the resulting map from $W_m(k_0)$ an isomorphism. This is also an isomorphism of solid modules: the same finite filtration applies on extremally disconnected profinite parameters. The argument over $\overline{\mathbb{F}}_p$ proves the corresponding rank-one assertion for T_m in the statement.

To compute the action before extending constants, choose C with the specified embedding of k_0 into C^b . By Lemma C.1, the completed source frame field in the k_0 -model is the completed perfection of a Laurent series field over k_0 , so it satisfies the field hypothesis of [2, Corollary 3.6.8]. The natural pullback of [2, Corollary 3.6.8 and the proof of Proposition 3.6.9] gives, at length one, Frobenius descent from the source perfectoid cover. At every finite Witt length the same pullback is an equivalence by induction through the additive Verschiebung sequences: it is a morphism of their exact triangles, and both pullback and φ_q -fixed points respect these triangles. Taking the commuting S_3 -invariants gives the natural equivalence

$$R\Gamma(\mathrm{BC}_{k_0}(-1/2)^\times, W_m \mathcal{O}^b) \simeq R\Gamma(\mathrm{BC}_{k_0}(-1/2)^\times \times_{\mathrm{Spd} k_0} \mathrm{Spd} C, W_m \mathcal{O}^b)^{h\varphi_q}. \quad (\mathrm{G.9})$$

Here φ_q acts on the C factor and fixes k_0 . Proposition 3.1.2 of [2] identifies the base-changed punctured space with $\mathbb{H}_{F,C}^{1,\diamond}/F$, compatibly with $\mathbb{Q}_p^\times$ -dilation. The right-hand complex before φ_q -fixed points has degree-two cohomology zero by (G.8). The two-term fixed-point sequence for φ_q therefore gives an injection

$$T_m(k_0) \hookrightarrow \mathcal{H}^3 R\Gamma(\mathbb{H}_{F,C}^{1,\diamond}/F, W_m \mathcal{O}^b).$$

All these maps commute with dilation. Its scalar a^{-2} on the target, computed in Lemma G.4, belongs to $\mathbb{Z}/p^m$ and is fixed by Frobenius. The injection proves that the scalar on $T_m(k_0)$ is also a^{-2} ; no choice of an integral Frobenius basis for the trace line is required.

The natural pullback to $\overline{\mathbb{F}}_p$ gives

$$W_m(\overline{\mathbb{F}}_p) \otimes_{W_m(k_0)} T_m(k_0) \longrightarrow T_m.$$

Its residue is the canonical base-change isomorphism of Lemma D.11. That lemma verifies the completed scalar extension on the actual frame-field nerve, using Lemma D.10; in particular no general exactness of completion is assumed. Both modules are free of rank one at every finite length. Pulling back a lifted residue generator and applying nilpotent Nakayama therefore proves that the displayed map is an isomorphism. It is the actual geometric pullback, so it transports the full scalar a^{-2} to T_m .

Finally we identify this dilation with the source norm action. With the fixed determinant normalization, the actual extension action derived in the proof of Lemma 2.7 is

$$e \mapsto \frac{\mathrm{Nrd}_3(g_3)}{\mathrm{Nrd}_2(g_2)}(g_2^{-1})^*e.$$

For target component $g_2 = 1$, a source element of norm a acts by $e \mapsto ae$. Every $a \in \mathbb{Z}_p^\times$ has a source representative, since the reduced norm on $\mathcal{O}_{D_3}^\times$ is surjective. For example this follows by restricting to the units of an unramified cubic maximal subfield: its residue norm is surjective, and its trace on successive principal-unit quotients is surjective. The induced action on the S_3 -quotient depends only on a ; no choice of a central representative is needed. Proposition 3.1.2 of [2] sends that pure dilation to $z \mapsto az$, with the action convention fixed above. This proves the asserted actual source character on T_m .

It remains to pass to ASW. Surjectivity of length reduction permits compatible bases of the T_m . Their inverse limit is a free $W(\overline{\mathbb{F}}_p)$ -line, with no derived inverse-limit correction. Frobenius is bijective and semilinear on this line. Over the algebraically closed residue field, a Frobenius-fixed residue basis exists; at each further Witt length the correction equation is an Artin–Schreier equation in $\overline{\mathbb{F}}_p$. Solving these equations successively gives compatible Frobenius-fixed bases. Thus $\varphi - 1$ is surjective on T_m and its kernel is a free $\mathbb{Z}/p^m$ -line. Surjectivity holds as a solid morphism as well: sections on a profinite parameter have finitely many values at each finite length, so their preimages can be chosen on clopen pieces.

The constant term has the same ASW calculation, with specified kernel $\mathbb{Z}/p^m$. Taking the fiber of $\varphi - 1$ in the constant-unit triangle leaves only its upper kernel in degree three. Its length reductions are surjective. Derived inverse limits therefore identify the ASW unit cofiber with a free $\mathbb{Z}_p$ -line in degree three. All the source scalars a^{-2} belong to $\mathbb{Z}_p$, hence commute with both Frobenius operations and remain the scalars on this ASW line. The stated values at -1 , at 5 when $p = 2$, and at $1 + p$ when p is odd follow by substitution. $\square$

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