

UNITS OF RING SPECTRA IN A FINITE POSTNIKOV RANGE

ABSTRACT. Let p be a prime and $n \geq 1$. We give a proof of the natural equivalence between the Postnikov slices of a p -local $\mathbb{E}_\infty$ -ring spectrum and its spectrum of units in degrees $n \leq i < pn$. The proof uses the Goodwillie tower of the reduced suspension spectrum of an infinite loop space. Tate vanishing splits the first $p - 1$ stages and supplies a section of the truncated stabilization counit. Testing this section on truncated sphere spectra identifies its effect on homotopy groups, allowing a based space equivalence to be replaced by a spectrum equivalence in the stated range. We also record the factorial-invertibility variant and identify the induced comparison on finite complexes with the truncated logarithm. Counterexamples for spectra of units show that the upper endpoint cannot be included uniformly.

1. THE COMPARISON THEOREM

For an $\mathbb{E}_\infty$ -ring spectrum R , its space of units $\mathrm{GL}_1 R$ is the union of the components of $\Omega^\infty R$ indexed by $(\pi_0 R)^\times$. Multiplication makes $\mathrm{GL}_1 R$ a grouplike $\mathbb{E}_\infty$ -space; the corresponding connective spectrum is denoted $\mathrm{gl}_1 R$. For $i > 0$ there are natural identifications $\pi_i \mathrm{gl}_1 R \cong \pi_i R$. The question is whether these identifications arise from an equivalence of spectra on a specified Postnikov interval.

We write $\mathrm{Sp}_{\geq a}$ for spectra whose homotopy groups vanish in degrees less than a , and use the convention

$$\tau_{[a,b]} E = \tau_{\leq b-1} \tau_{\geq a} E.$$

The subscript (p) denotes localization at the prime p . For integers $a \leq b$, $\mathrm{Sp}_{(p),[a,b]}$ is the full subcategory of p -local spectra whose homotopy groups are concentrated in $[a, b]$. We work in ∞ -categories. The category of based spaces is denoted $\mathcal{S}_*$, Σ^∞ is the reduced suspension spectrum functor, Σ_+^∞ first adjoins a disjoint basepoint, and $\otimes$ is the smash product of spectra.

Theorem 1.1. *Let p be a prime and $n \geq 1$. There is an equivalence of spectra*

$$\tau_{[n,pn]} \mathrm{gl}_1 R \simeq \tau_{[n,pn]} R,$$

natural in the p -local $\mathbb{E}_\infty$ -ring spectrum R . On each homotopy group in this interval, the equivalence induces the usual identification $\pi_i \mathrm{gl}_1 R \cong \pi_i R$.

The interval in Theorem 1.1 is the one proposed by Mathew and Stojanoska in [7, Remark 5.2.6], under the more general assumption that $(p - 1)!$ is invertible in $\pi_0 R$. Their Corollary 5.2.3 establishes the interval $[n, 2n)$ for $n \geq 2$. Heuts's seminar announcement [2] describes joint work with Clausen, Mathew, and Stojanoska on finite-range comparisons and truncated logarithms. These notes present an argument from the Goodwillie–Tate description of $\Sigma^\infty \Omega^\infty$.

On a finite based $(n - 1)$ -connected complex of dimension less than pn , the equivalence constructed here induces the finite logarithm $1 + x \mapsto \sum_{k=1}^{p-1} (-1)^{k-1} x^k / k$; see Corollary A.2. Proposition 5.2 gives a units-spectrum counterexample to including degree pn , using the obstruction of Heard, Mathew, and Stojanoska [3, Lemma 7.6].

Fix p and n throughout the notes. The comparison of units follows from the following statement.

Proposition 1.2. *For $E, F \in \mathrm{Sp}_{(p),[n,pn-1]}$, there is a map of mapping spaces*

$$\mathrm{Lin}_{E,F} : \mathrm{Map}_{\mathcal{S}_*}(\Omega^\infty E, \Omega^\infty F) \longrightarrow \mathrm{Map}_{\mathrm{Sp}_{(p)}}(E, F),$$

natural with respect to spectrum maps in E and F , such that

$$\pi_i \operatorname{Lin}(f) = \pi_i f \quad (n \leq i < pn).$$

Consequently, a based space equivalence f gives a spectrum equivalence $\operatorname{Lin}(f)$. If $g : E \rightarrow F$ is a spectrum map, then $\operatorname{Lin}(\Omega^\infty g) \simeq g$.

Here naturality refers to maps of spectra between the source and target objects; no composition law for arbitrary based space maps is part of the statement. To prove the proposition, we construct a natural section of the truncated counit $\tau_{\leq pn-1} \Sigma_{(p)}^\infty \Omega^\infty E \rightarrow E$. Truncated sphere spectra then show that the section induces ordinary stabilization on homotopy groups. This identifies the homotopy-group action of the spectrum map obtained by composing the section, the suspension of f , and the counit. Theorem 1.1 follows by applying that construction to translation by -1 on the units space.

2. A SECTION OF THE TRUNCATED COUNIT

Consider the functor

$$\Sigma_{(p)}^\infty \Omega^\infty : \operatorname{Sp}_{(p)} \longrightarrow \operatorname{Sp}_{(p)}.$$

Its Goodwillie tower has stages $P_d(\Sigma_{(p)}^\infty \Omega^\infty)$. The homogeneous layer $D_d(\Sigma_{(p)}^\infty \Omega^\infty)$ is the fiber of the projection from the d th stage to the $(d-1)$ st stage. At each finite stage, localization gives

$$P_d(\Sigma_{(p)}^\infty \Omega^\infty)(E) \simeq (P_d(\Sigma^\infty \Omega^\infty)(E))_{(p)}.$$

Indeed, p -local spectra are closed under finite colimits, and localization preserves the finite limits and filtered colimits in the construction of each Goodwillie stage. Convergence of the localized tower is addressed in Lemma 2.2.

2.1. The Goodwillie–Tate square. The required properties of this tower are

$$\begin{aligned} P_1(\Sigma_{(p)}^\infty \Omega^\infty)(E) &\simeq E, \\ D_d(\Sigma_{(p)}^\infty \Omega^\infty)(E) &\simeq (E^{\otimes d})_{h\Sigma_d}. \end{aligned} \tag{1}$$

The map $\Sigma_{(p)}^\infty \Omega^\infty E \rightarrow P_1(\Sigma_{(p)}^\infty \Omega^\infty)(E)$ is the stabilization counit. The tower converges to $\Sigma_{(p)}^\infty \Omega^\infty E$ for $E \in \operatorname{Sp}_{\geq 1}$. These facts are stated in [5, §6.2, pp. 264–265].

The McCarthy–Kuhn description gives a natural homotopy pullback square

$$\begin{array}{ccc} P_d(\Sigma_{(p)}^\infty \Omega^\infty)(E) & \longrightarrow & (E^{\otimes d})^{h\Sigma_d} \\ \downarrow & & \downarrow \\ P_{d-1}(\Sigma_{(p)}^\infty \Omega^\infty)(E) & \longrightarrow & (E^{\otimes d})^{t\Sigma_d}. \end{array} \tag{2}$$

The general square of [4, Proposition 1.9 and §5] uses the diagonal of the multilinearized cross-effect. For $\Sigma_{(p)}^\infty \Omega^\infty$ this is $E^{\otimes d}$ with its permutation action, giving (2); see [5, §6.2, p. 265].

Here Y^{tG} denotes the cofiber of the norm map $Y_{hG} \rightarrow Y^{hG}$ for a finite group G acting on a spectrum Y .

Lemma 2.1. *If G is a finite group of order prime to p and Y is a p -local spectrum with G -action, then*

$$Y^{tG} \simeq 0. \tag{3}$$

Proof. For a finite group, induction and coinduction from the trivial subgroup agree in spectra. Their adjunction maps give

$$Y \xrightarrow{u} \operatorname{Ind}_{\{1\}}^G \operatorname{Res}_{\{1\}}^G Y \xrightarrow{v} Y.$$

After forgetting the action, vu is $|G|\operatorname{id}$ and hence an equivalence. Forgetting the action detects equivalences, so vu is an equivalence of G -spectra. Thus $u(vu)^{-1}$ is a section of v ,

exhibiting Y as a retract of an induced object. The Tate construction vanishes on induced objects and preserves retracts. $\square$

For $d < p$, Lemma 2.1 applies to $G = \Sigma_d$. The pullback square and the norm equivalence therefore give

$$\begin{aligned} P_d(\Sigma_{(p)}^\infty \Omega^\infty)(E) &\simeq P_{d-1}(\Sigma_{(p)}^\infty \Omega^\infty)(E) \oplus (E^{\otimes d})^{h\Sigma_d} \\ &\simeq P_{d-1}(\Sigma_{(p)}^\infty \Omega^\infty)(E) \oplus (E^{\otimes d})_{h\Sigma_d}. \end{aligned}$$

Iterating these equivalences yields

$$P_{p-1}(\Sigma_{(p)}^\infty \Omega^\infty)(E) \simeq \bigoplus_{d=1}^{p-1} (E^{\otimes d})_{h\Sigma_d}. \quad (4)$$

The projection to the first summand is the tower map to $P_1(\Sigma_{(p)}^\infty \Omega^\infty)(E)$. Its section will be denoted

$$j_E : E \longrightarrow P_{p-1}(\Sigma_{(p)}^\infty \Omega^\infty)(E). \quad (5)$$

The maps from the upper-left vertex of (2) to its two adjacent vertices define a natural equivalence to their product when the lower-right vertex vanishes. Its inverse and the inclusion with second coordinate zero define the section in the functor ∞ -category. Hence j_E is a coherent natural transformation.

2.2. The Postnikov cutoff. For an n -connective spectrum E , the layers satisfy

$$(E^{\otimes d})_{h\Sigma_d} \in \mathrm{Sp}_{\geq dn}. \quad (6)$$

Smash products add connectivity, and homotopy orbits preserve this lower bound. The layers with $d \geq p$ therefore start in degree pn .

Lemma 2.2. *For $E \in \mathrm{Sp}_{(p), \geq n}$, the tower map induces an equivalence*

$$\tau_{\leq pn-1} \Sigma_{(p)}^\infty \Omega^\infty E \xrightarrow{\sim} \tau_{\leq pn-1} P_{p-1}(\Sigma_{(p)}^\infty \Omega^\infty)(E). \quad (7)$$

Proof. By (6), all maps $P_d(\Sigma_{(p)}^\infty \Omega^\infty)(E) \rightarrow P_{d-1}(\Sigma_{(p)}^\infty \Omega^\infty)(E)$ with $d \geq p$ induce isomorphisms on π_i for $i < pn$. For every fixed integer k , the groups $\pi_k P_d(\Sigma_{(p)}^\infty \Omega^\infty)(E)$ are eventually constant as d increases. Thus the Milnor term $\lim_d^1 \pi_{i+1} P_d(\Sigma_{(p)}^\infty \Omega^\infty)(E)$ vanishes, including when $i = pn - 1$. Convergence identifies the limit with $\Sigma_{(p)}^\infty \Omega^\infty E$ and proves the claim.

Convergence passes to the localized tower by the same bounds. Termwise localization of the integral tower preserves the layers, and both homotopy-group towers are eventually constant in each degree. Localizing their eventual values identifies the localized integral limit with the limit of the localized tower. $\square$

For $E \in \mathrm{Sp}_{(p), [n, pn-1]}$, truncate (5) and compose with the inverse of (7) to obtain

$$s_E : E \longrightarrow \tau_{\leq pn-1} \Sigma_{(p)}^\infty \Omega^\infty E. \quad (8)$$

Since E is $(pn - 1)$ -truncated, the counit induces

$$c_E : \tau_{\leq pn-1} \Sigma_{(p)}^\infty \Omega^\infty E \longrightarrow E. \quad (9)$$

Compatibility with the projection to $P_1(\Sigma_{(p)}^\infty \Omega^\infty)(E)$ gives

$$c_E s_E \simeq \mathrm{id}_E. \quad (10)$$

The inverse in (7) is taken in the functor ∞ -category, so s and c are natural transformations on $\mathrm{Sp}_{(p), [n, pn-1]}$.

More generally, (4) and Lemma 2.2 give the natural decomposition

$$\tau_{\leq pn-1} \Sigma_{(p)}^\infty \Omega^\infty E \simeq E \oplus \bigoplus_{d=2}^{p-1} \tau_{\leq pn-1} (E^{\otimes d})_{h\Sigma_d}.$$

The maps s_E and c_E are the inclusion and projection of the first summand.

3. THE HOMOTOPY GROUPS OF THE SECTION

The retraction identity alone does not identify the homotopy-group action of s_E with the stabilization of elements of $\pi_i \Omega^\infty E$. This identification is what allows a space equivalence to give a spectrum equivalence.

For $n \leq i < pn$, write

$$\sigma_E : \pi_i E = \pi_i \Omega^\infty E \longrightarrow \pi_i(\tau_{\leq pn-1} \Sigma_{(p)}^\infty \Omega^\infty E)$$

for ordinary stabilization followed by truncation. The adjunction triangle identity gives

$$c_{E*} \sigma_E = \text{id}. \quad (11)$$

Lemma 3.1. *For $E \in \text{Sp}_{(p),[n,pn-1]}$ and $n \leq i < pn$,*

$$s_{E*} = \sigma_E : \pi_i E \longrightarrow \pi_i(\tau_{\leq pn-1} \Sigma_{(p)}^\infty \Omega^\infty E). \quad (12)$$

Proof. Consider the truncated sphere spectrum

$$A_i = \tau_{\leq pn-1} \Sigma_{(p)}^i \mathbb{S}.$$

It belongs to $\text{Sp}_{(p),[n,pn-1]}$ and is i -connective. Its nonlinear Goodwillie layers start in degree $2i > i$, so the counit induces an isomorphism $c_{A_i*} : \pi_i(\tau_{\leq pn-1} \Sigma_{(p)}^\infty \Omega^\infty A_i) \rightarrow \pi_i A_i$. Equations (10) and (11) identify two right inverses to this isomorphism. Therefore

$$s_{A_i*} = \sigma_{A_i} \quad \text{on } \pi_i A_i. \quad (13)$$

Let $x \in \pi_i E$. Since E is $(pn-1)$ -truncated, the truncation adjunction on homotopy classes of spectrum maps gives

$$[A_i, E] \cong [\Sigma^i \mathbb{S}_{(p)}, E] = \pi_i E.$$

Choose the map $a : A_i \rightarrow E$ corresponding to x , and let $1 \in \pi_i A_i \cong \mathbb{Z}_{(p)}$ be the sphere generator. Naturality of s and of stabilization gives

$$\begin{aligned} s_{E*}(x) &= (\tau_{\leq pn-1} \Sigma_{(p)}^\infty \Omega^\infty a)_* s_{A_i*}(1) \\ &= (\tau_{\leq pn-1} \Sigma_{(p)}^\infty \Omega^\infty a)_* \sigma_{A_i}(1) = \sigma_E(x). \end{aligned}$$

□

Proof of Proposition 1.2. For a based space map $f : \Omega^\infty E \rightarrow \Omega^\infty F$, define

$$\text{Lin}(f) = c_F \circ (\tau_{\leq pn-1} \Sigma_{(p)}^\infty f) \circ s_E : E \longrightarrow F. \quad (14)$$

Functoriality of $\tau_{\leq pn-1} \Sigma_{(p)}^\infty$ and composition with s_E and c_F make this a map of mapping spaces. Lemma 3.1, naturality of stabilization for based space maps, and (11) imply

$$\begin{aligned} \text{Lin}(f)_* &= c_{F*}(\tau_{\leq pn-1} \Sigma_{(p)}^\infty f)_* \sigma_E \\ &= c_{F*} \sigma_F f_* = f_*. \end{aligned} \quad (15)$$

Both spectra have homotopy groups only in $[n, pn-1]$, so a space equivalence f yields a spectrum equivalence $\text{Lin}(f)$.

For a spectrum map $g : E \rightarrow F$, naturality of the counit gives

$$\text{Lin}(\Omega^\infty g) = c_F(\tau_{\leq pn-1} \Sigma_{(p)}^\infty \Omega^\infty g) s_E \simeq g c_E s_E \simeq g. \quad (16)$$

For spectrum maps $a : E' \rightarrow E$ and $b : F \rightarrow F'$, naturality of s and c likewise gives

$$\text{Lin}((\Omega^\infty b) f (\Omega^\infty a)) \simeq b \text{Lin}(f) a.$$

These identities are composites of natural transformations in the functor ∞ -category, which supplies their higher coherences. □

Corollary 3.2. *For $E, F \in \mathrm{Sp}_{(p),[n,pn-1]}$, there is a natural equivalence*

$$\mathrm{Map}_{\mathcal{S}_*}(\Omega^\infty E, \Omega^\infty F) \simeq \prod_{d=1}^{p-1} \mathrm{Map}_{\mathrm{Sp}_{(p)}}(\tau_{\leq pn-1}(E^{\otimes d})_{h\Sigma_d}, F).$$

Under this equivalence, $\mathrm{Lin}_{E,F}$ is projection to the $d = 1$ factor, and Ω^∞ on spectrum maps is inclusion of that factor with all other coordinates zero.

Proof. Suspension-spectrum adjunction identifies the left side with $\mathrm{Map}_{\mathrm{Sp}_{(p)}}(\Sigma_{(p)}^\infty \Omega^\infty E, F)$. Since F is $(pn - 1)$ -truncated, the source can be truncated at $pn - 1$. The decomposition above then gives the product. The descriptions of Lin and Ω^∞ follow from (14) and the first-summand descriptions of s_E and c_E . $\square$

4. THE SPECTRUM OF UNITS

Proof of Theorem 1.1. The spectra $\tau_{[n,pn]} \mathrm{gl}_1 R$ and $\tau_{[n,pn]} R$ belong to $\mathrm{Sp}_{(p),[n,pn-1]}$: the positive homotopy groups of $\mathrm{gl}_1 R$ are those of the p -local spectrum R , and $n \geq 1$.

The inclusion $\mathrm{GL}_1 R \subseteq \Omega^\infty R$ followed by translation by -1 restricts to a natural based equivalence

$$(\mathrm{GL}_1 R)_1 \xrightarrow{\sim} (\Omega^\infty R)_0, \quad u \mapsto u - 1. \quad (17)$$

The subscripts denote connected components. This comparison is used in [7, §2.2 and the proof of Corollary 5.2.3]. Functorial $(n - 1)$ -connected covers and Postnikov truncations through degree $pn - 1$ of based spaces identify with Ω^∞ of the corresponding spectrum slices by the Postnikov maps. Applied to (17), they give

$$f_R : \Omega^\infty \tau_{[n,pn]} \mathrm{gl}_1 R \xrightarrow{\sim} \Omega^\infty \tau_{[n,pn]} R. \quad (18)$$

This construction depends only on the positive Postnikov slices, so it also applies to nonconnective R .

Proposition 1.2 gives

$$\lambda_R = \mathrm{Lin}(f_R) : \tau_{[n,pn]} \mathrm{gl}_1 R \xrightarrow{\sim} \tau_{[n,pn]} R. \quad (19)$$

Its homotopy-group action is the usual identification, by (15) and the construction of f_R . For a ring map $\varphi : R \rightarrow R'$, naturality of (18) and Proposition 1.2 give the commutative square

$$\begin{array}{ccc} \tau_{[n,pn]} \mathrm{gl}_1 R & \xrightarrow{\lambda_R} & \tau_{[n,pn]} R \\ \tau_{[n,pn]} \mathrm{gl}_1(\varphi) \downarrow & & \downarrow \tau_{[n,pn]} \varphi \\ \tau_{[n,pn]} \mathrm{gl}_1 R' & \xrightarrow{\lambda_{R'}} & \tau_{[n,pn]} R'. \end{array}$$

Its homotopies and higher coherences are induced by those of f_R and of the natural transformations defining Lin . $\square$

Corollary 4.1. *Let p be prime and $n \geq 1$. The conclusion of Theorem 1.1 holds for $\mathbb{E}_\infty$ -ring spectra R such that $(p - 1)!$ is invertible in $\pi_0 R$.*

Proof. Work throughout in $\mathbb{S}[1/(p - 1)!]$ -local spectra. Multiplication by each divisor of $(p - 1)!$ is an equivalence on R , since it is multiplication by a unit in $\pi_0 R$; the same is true on the positive slice of $\mathrm{gl}_1 R$. For every $d < p$, the order $d!$ of Σ_d is invertible in this localized category. Thus Lemma 2.1, the finite-stage splitting, and the connectivity argument apply without change. $\square$

For $p = 2$, the construction uses only $P_1(\Sigma_{(p)}^\infty \Omega^\infty)$ and gives the interval $[n, 2n)$. The hypothesis $n \geq 1$ is used both for tower convergence and to discard $\pi_0 \mathrm{gl}_1 R$, which need not be a localized abelian group even when the positive homotopy groups are.

5. THE UPPER ENDPOINT

Example 5.1. Suppose that p is odd and $n = 2a \geq 2$. There are p -local spectra with homotopy groups only in $[n, pn]$ whose underlying infinite loop spaces are equivalent as based spaces, but which are not equivalent as spectra.

Proof. Let β denote the mod- p Bockstein and P^a the Steenrod reduced power. Consider the nonzero stable operation

$$\theta = \beta P^a : \Sigma^n H\mathbb{F}_p \longrightarrow \Sigma^{pn+1} H\mathbb{F}_p. \quad (20)$$

Its degree is $2a(p-1) + 1 = pn + 1 - n$. Nonvanishing follows from the admissible basis of the Steenrod algebra; see [1, §4.L, pp. 499–500]. For the fundamental class $u \in H^n(K(\mathbb{F}_p, n); \mathbb{F}_p)$, the top-power formula and the Bockstein derivation identity give

$$P^a(u) = u^p, \quad \beta P^a(u) = \beta(u^p) = p u^{p-1} \beta(u) = 0.$$

These identities are recorded in [1, §3.E, p. 304; §4.L, p. 490]. It follows that $\Omega^\infty \theta$ is nullhomotopic.

Put

$$V = \text{fib}(\theta), \quad W = \Sigma^n H\mathbb{F}_p \oplus \Sigma^{pn} H\mathbb{F}_p.$$

A choice of nullhomotopy gives

$$\Omega^\infty V \simeq K(\mathbb{F}_p, n) \times K(\mathbb{F}_p, pn) \simeq \Omega^\infty W.$$

The stable Postnikov invariant of V is the nonzero map θ , whereas the corresponding invariant of W is zero. Hence $V \not\simeq W$. $\square$

This example prevents a uniform extension of Proposition 1.2 to spectra concentrated in $[n, pn]$. For spectra of units, the same endpoint is obstructed by the multiplicative argument in [3, Lemma 7.6].

Proposition 5.2. *Let p be odd and let $n \geq 2$ be even. If R is a connective commutative $H\mathbb{F}_p$ -algebra with*

$$\pi_* R \cong \mathbb{F}_p[x]/(x^{p+1}), \quad |x| = n,$$

then

$$\tau_{[n, pn+1)} \text{gl}_1 R \not\simeq \tau_{[n, pn+1)} R.$$

Such an algebra is given by $\tau_{\leq pn} \text{Sym}_{H\mathbb{F}_p}(\Sigma^n H\mathbb{F}_p)$, where $\text{Sym}_{H\mathbb{F}_p}$ is the free commutative $H\mathbb{F}_p$ -algebra functor.

Proof. The underlying $H\mathbb{F}_p$ -module splits as

$$\tau_{[n, pn+1)} R \simeq \bigoplus_{k=1}^p \Sigma^{kn} H\mathbb{F}_p.$$

If the two slices were equivalent, the degree- n summand would give a spectrum map $\Sigma^n H\mathbb{F}_p \rightarrow \text{gl}_1 R$ nonzero on π_n . Here the units slice maps to $\text{gl}_1 R$ because R , and hence $\text{gl}_1 R$, is pn -truncated. The units adjunction and extension of scalars along $\mathbb{S} \rightarrow H\mathbb{F}_p$ would then give a map of commutative $H\mathbb{F}_p$ -algebras

$$H\mathbb{F}_p \otimes_{\mathbb{S}} \Sigma_+^\infty K(\mathbb{F}_p, n) \longrightarrow R.$$

The bottom Hurewicz class $z \in H_n(K(\mathbb{F}_p, n); \mathbb{F}_p)$ maps to ax for some $a \in \mathbb{F}_p^\times$. Under the Pontryagin product induced by addition on $K(\mathbb{F}_p, n)$, one has $z^p = 0$. This is the homology calculation used in [3, proof of Lemma 7.6]. Multiplicativity would imply

$$0 = (ax)^p = a^p x^p,$$

contradicting $x^p \neq 0$.

To construct an algebra with the stated homotopy ring, the underlying module of the free commutative $H\mathbb{F}_p$ -algebra is

$$\mathrm{Sym}_{H\mathbb{F}_p}(\Sigma^n H\mathbb{F}_p) \simeq \bigoplus_{k \geq 0} ((\Sigma^n H\mathbb{F}_p)^{\otimes_{H\mathbb{F}_p} k})_{h\Sigma_k}.$$

Since n is even, the permutation action on each tensor power is trivial. The k th homotopy-orbit summand therefore has i th homotopy group $H_{i-kn}(\Sigma_k; \mathbb{F}_p)$. For $k < p$, positive group homology vanishes because $k!$ is invertible in $\mathbb{F}_p$. For $k = p$, only H_0 remains in degrees at most pn ; the terms with $k > p$ start above pn . Multiplication on the bottom classes is induced by block inclusions of symmetric groups and sends the generator of H_0 to the generator. Truncation at pn therefore gives the displayed polynomial quotient, as in the construction preceding [3, Lemma 7.6]. $\square$

At $p = 2$, [7, Example 5.2.1 and Proposition 5.2.2] treats the ring spectrum associated to $\mathbb{F}_2[x]/(x^3)$ with $|x| = n$, and identifies the nonzero invariant of its units spectrum as $\mathrm{Sq}^{n+1} : \Sigma^n H\mathbb{F}_2 \rightarrow \Sigma^{2n+1} H\mathbb{F}_2$.

APPENDIX A. LOGARITHMS AND RELATIVE UNITS

Throughout this appendix, R is a p -local $\mathbb{E}_\infty$ -ring spectrum.

A.1. The cohomological logarithm. Let A be a commutative ring in which $(p-1)!$ is invertible, and let $J \subseteq A$ be an ideal satisfying $J^p = 0$. The finite formulas

$$\log_{<p}(1+x) = \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} x^k, \quad \exp_{<p}(x) = \sum_{k=0}^{p-1} \frac{x^k}{k!}$$

give inverse group isomorphisms $(1+J, \times) \cong (J, +)$. Indeed, the formal logarithm and exponential identities in total degree less than p use only the displayed invertible denominators; all remaining terms vanish in J^p .

Let X be a finite based $(n-1)$ -connected CW complex with $\dim X < pn$. The smash power $X^{\wedge p}$ is $(pn-1)$ -connected. Hence the reduced diagonal $X \rightarrow X^{\wedge p}$ is nullhomotopic by the dimension bound. Products of p reduced cohomology classes factor through this diagonal. The augmentation ideal

$$\tilde{R}^0(X) = \ker(R^0(X) \rightarrow \pi_0 R)$$

therefore satisfies $(\tilde{R}^0(X))^p = 0$. The group $\widetilde{\mathrm{gl}_1 R}^0(X)$ identifies with $1 + \tilde{R}^0(X)$ under multiplication, so the logarithm gives

$$\widetilde{\mathrm{gl}_1 R}^0(X) \cong \tilde{R}^0(X).$$

Here reduced degree-zero cohomology is represented by based maps from X to the relevant infinite loop space. Corollary A.2 identifies this logarithm with the map induced by (19).

A.2. Relative units in a Postnikov extension. For the Postnikov map

$$\tau_{\leq pn-1} \tau_{\geq 0} R \rightarrow \tau_{\leq n-1} \tau_{\geq 0} R$$

of connective $\mathbb{E}_\infty$ -rings, define the spectrum fibers

$$\begin{aligned} J &= \mathrm{fib}(\tau_{\leq pn-1} \tau_{\geq 0} R \rightarrow \tau_{\leq n-1} \tau_{\geq 0} R), \\ U &= \mathrm{fib}(\mathrm{gl}_1(\tau_{\leq pn-1} \tau_{\geq 0} R) \rightarrow \mathrm{gl}_1(\tau_{\leq n-1} \tau_{\geq 0} R)). \end{aligned}$$

The Postnikov maps identify

$$J \simeq \tau_{[n, pn]} R, \quad U \simeq \tau_{[n, pn]} \mathrm{gl}_1 R.$$

For the second identification, the map on units induces an isomorphism on π_0 and the same positive-degree homotopy groups as the Postnikov map. Moreover,

$$J \in \mathrm{Sp}_{\geq n}, \quad \tau_{\leq pn-1} J^{\otimes p} = 0.$$

The underlying space of U is identified with that of J by $u \mapsto u - 1$. In these coordinates its multiplicative group law is $x \star y = x + y + xy$. Proposition 1.2 applied to this space equivalence gives $U \simeq J$.

A.3. The convolution logarithm. For $E \in \mathrm{Sp}_{(p),[n,pn-1]}$, set

$$H_E = \Sigma_{(p),+}^\infty \Omega^\infty E.$$

Addition and the diagonal on $\Omega^\infty E$ induce maps

$$\mu : H_E \otimes H_E \longrightarrow H_E, \quad \Delta : H_E \longrightarrow H_E \otimes H_E.$$

The zero point and the map to a point induce the unit and augmentation

$$\eta : \mathbb{S}_{(p)} \longrightarrow H_E, \quad \varepsilon : H_E \longrightarrow \mathbb{S}_{(p)}.$$

For spectrum endomorphisms $f, g : H_E \rightarrow H_E$, convolution is $f * g = \mu \circ (f \otimes g) \circ \Delta$. The finite expression

$$e = \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} (\mathrm{id} - \eta\varepsilon)^{*k} \quad (21)$$

is the convolution form of a truncated logarithm. The splitting $\varepsilon\eta = \mathrm{id}$ identifies the augmentation ideal $\mathrm{fib}(\varepsilon)$ with $\Sigma_{(p)}^\infty \Omega^\infty E$, which is n -connective. The p th convolution power of $\mathrm{id} - \eta\varepsilon$ factors through $(\Sigma_{(p)}^\infty \Omega^\infty E)^{\otimes p}$, so its composite with $H_E \rightarrow \tau_{\leq pn-1} H_E$ is zero.

Proposition A.1. *The endomorphism e preserves the augmentation ideal and satisfies*

$$\tau_{\leq pn-1}(e|_{\Sigma_{(p)}^\infty \Omega^\infty E}) \simeq s_{EC} e.$$

Thus the truncated convolution logarithm is projection onto the linear summand of the natural Goodwillie decomposition.

Proof. For an integer $r \geq 0$, let

$$\Psi_r = \Sigma_{(p),+}^\infty \Omega^\infty (r \mathrm{id}_E) : H_E \longrightarrow H_E.$$

Addition on $\Omega^\infty E$ gives $\Psi_0 = \eta\varepsilon$ and $\mathrm{id}^{*r} = \Psi_r$, with the zeroth convolution power understood to be the convolution unit $\eta\varepsilon$. Bilinearity of convolution gives

$$(\mathrm{id} - \eta\varepsilon)^{*k} = \sum_{r=0}^k (-1)^{k-r} \binom{k}{r} \Psi_r.$$

Every positive convolution power of $\mathrm{id} - \eta\varepsilon$ kills the unit and has zero augmentation, so e restricts to the augmentation ideal.

The natural splitting (4), followed by Lemma 2.2, identifies the truncated restriction of Ψ_r with the direct sum of

$$\tau_{\leq pn-1}((r \mathrm{id}_E)^{\otimes d})_{h\Sigma_d}, \quad 1 \leq d < p.$$

These maps are $r^d \mathrm{id}$. Multilinearity gives the identity after forgetting the Σ_d -action. To recover it equivariantly, restriction on homotopy classes is injective when $d!$ is invertible: the source is a retract of an induced object by the proof of Lemma 2.1, and maps from that induced object are determined by adjunction. The same argument applies in the spectrum-valued functor category. Thus naturality of the splitting makes Ψ_r diagonal, with scalar r^d on its d th summand.

Consequently, e acts on that summand by

$$\sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} \sum_{r=0}^k (-1)^{k-r} \binom{k}{r} r^d = \sum_{k=1}^d (-1)^{k-1} (k-1)! S(d, k),$$

where $S(d, k)$ is the number of partitions of a d -element set into k nonempty subsets. The equality is the finite-difference identity $\sum_{r=0}^k (-1)^{k-r} \binom{k}{r} r^d = k! S(d, k)$. Comparing coefficients of $t^d/d!$ in

$$\sum_{k \geq 1} \frac{(-1)^{k-1}}{k} (\exp t - 1)^k = t$$

shows that the displayed scalar is 1 for $d = 1$ and 0 for $2 \leq d < p$. The first-summand inclusion and projection are s_E and c_E , which proves the identity. $\square$

Corollary A.2. *Let X be a finite based $(n-1)$ -connected CW complex with $\dim X < pn$. The equivalence λ_R induces*

$$(\lambda_R)_X : \widetilde{\mathrm{gl}}_1^0 R(X) \longrightarrow \widetilde{R}^0(X), \quad (\lambda_R)_X(1+x) = \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} x^k.$$

The same formula holds for the comparison of Corollary 4.1 when $(p-1)!$ is invertible in $\pi_0 R$.

Proof. There are natural identifications of homotopy classes

$$\begin{aligned} \widetilde{\mathrm{gl}}_1^0 R(X) &\cong [\Sigma_{(p)}^\infty X, \tau_{[n, pn]} \mathrm{gl}_1 R], \\ \widetilde{R}^0(X) &\cong [\Sigma_{(p)}^\infty X, \tau_{[n, pn]} R]. \end{aligned}$$

For the lower covers, use the n -connectivity of $\Sigma^\infty X$ and Postnikov orthogonality. For the upper truncations, cellular dimension gives $[\Sigma^\infty X, K] = [\Sigma^\infty X, \Sigma K] = 0$ for every pn -connective spectrum K . This makes the upper Postnikov maps induce bijections on homotopy classes. Localization of the source is taken after passing to the positive covers, which are p -local.

Represent a unit $u \in 1 + \widetilde{R}^0(X)$ by a map

$$a : \Sigma_{(p)}^\infty X \longrightarrow \tau_{[n, pn]} \mathrm{gl}_1 R.$$

Let b be the suspension of its based adjoint, followed by truncation of the target at $pn-1$, and let e be the convolution logarithm of the units slice, acting on its reduced suspension spectrum. The adjunction counit gives $a \simeq c_{\tau_{[n, pn]} \mathrm{gl}_1 R} b$. Proposition A.1 therefore gives

$$s_{\tau_{[n, pn]} \mathrm{gl}_1 R} a \simeq (\tau_{\leq pn-1} e) b.$$

Substitute the finite-difference expression for e into (14). Multiplication by r on the units spectrum corresponds to $u \mapsto u^r$; translation by -1 in (18) then gives the cohomology class $u^r - 1$. Hence

$$\begin{aligned} (\lambda_R)_X(u) &= \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} \sum_{r=0}^k (-1)^{k-r} \binom{k}{r} (u^r - 1) \\ &= \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} (u - 1)^k. \end{aligned}$$

The last equality is the binomial identity for $k \geq 1$. For the factorial-invertibility variant, replace p -localization throughout by $\mathbb{S}[1/(p-1)!]$ -localization. The argument uses only the inverses of $d!$ for $d < p$ and of k for $1 \leq k < p$. $\square$

APPENDIX B. A SECTION FROM SCALAR WEIGHTS

The section (5) can also be recovered from the action of scalar maps on the finite Goodwillie tower. Choose an integer q whose residue generates $\mathbb{F}_p^\times$, and put

$$\psi = P_{p-1}(\Sigma_{(p)}^\infty \Omega^\infty)(q \mathrm{id}).$$

This is a natural endomorphism of $P_{p-1}(\Sigma_{(p)}^\infty \Omega^\infty)$. On the k th homogeneous layer it acts by q^k for $1 \leq k < p$. To interpret this assertion equivariantly, first use multilinearity to identify

the underlying tensor-power map with $q^k \text{id}$. The restriction map from equivariant homotopy classes to underlying homotopy classes is injective when $k!$ is invertible: restriction followed by transfer multiplies an equivariant class by $k!$. This argument applies in the spectrum-valued functor category, so the scalar identity is natural in E .

Work in $\text{Fun}(\text{Sp}_{(p)}, \text{Sp}_{(p)})$. On the tower term $P_d(\Sigma_{(p)}^\infty \Omega^\infty)$, write $\psi_d = P_d(\Sigma_{(p)}^\infty \Omega^\infty)(q \text{id})$. We claim that

$$\prod_{k=1}^d (\psi_d - q^k) = 0 \quad \text{in the homotopy category,} \quad 1 \leq d < p.$$

For $d = 1$ this is the scalar action on $P_1(\Sigma_{(p)}^\infty \Omega^\infty) \simeq \text{id}$. For the induction step, the natural map $\psi_d - q^d$ restricts to zero on the fiber $D_d(\Sigma_{(p)}^\infty \Omega^\infty)$ and thus factors through the projection to $P_{d-1}(\Sigma_{(p)}^\infty \Omega^\infty)$. Composing on the right with $\prod_{k < d} (\psi_d - q^k)$ gives zero by naturality of that projection and the induction hypothesis. The factors commute, since they are polynomials in ψ_d . In particular,

$$\prod_{k=1}^{p-1} (\psi - q^k) = 0. \quad (22)$$

The scalars $q, q^2, \dots, q^{p-1}$ have distinct residues modulo p . Hence their pairwise differences are units in $\mathbb{Z}_{(p)}$, and polynomial interpolation defines

$$e_1 = \prod_{k=2}^{p-1} \frac{\psi - q^k}{q - q^k}. \quad (23)$$

The empty product for $p = 2$ is the identity. The denominators in this interpolation depend on p -localization. They need not be invertible after inverting $(p-1)!$: for $p = 5$ and $q = 2$, one has $q - q^4 = -14$, which is not a unit in $\mathbb{Z}[1/24]$. Equation (22) implies $e_1^2 = e_1$ in the homotopy category: the polynomial $e_1(t)^2 - e_1(t)$ vanishes at each q^k and is divisible by $\prod_{k=1}^{p-1} (t - q^k)$.

The stable functor category has countable colimits, so this homotopy idempotent splits; see [6, Lemma 1.2.4.6 and Warning 1.2.4.8]. Its image is naturally equivalent to $P_1(\Sigma_{(p)}^\infty \Omega^\infty)$. One can see this by applying the same polynomial to the entire finite tower and taking the telescope of that endomorphism. On $D_k(\Sigma_{(p)}^\infty \Omega^\infty)$ the polynomial acts as zero for $k \geq 2$, and on $P_1(\Sigma_{(p)}^\infty \Omega^\infty)$ it acts as the identity. Filtered colimits are exact in spectra, so the telescoped tower has zero higher fibers; its top term maps equivalently to $P_1(\Sigma_{(p)}^\infty \Omega^\infty)$. On the top term the telescope is the image of e_1 . The resulting retract therefore supplies a natural section of $P_{p-1}(\Sigma_{(p)}^\infty \Omega^\infty) \rightarrow P_1(\Sigma_{(p)}^\infty \Omega^\infty)$, as required in (5).

For example, when $p = 3$ and $q = 2$ the projector is $e_1 = (4 - \psi)/2$. At the next layer the congruence $q^p \equiv q \pmod{p}$ makes the linear and p th weights coincide. The denominator separating these two weights is then unavailable, just as the Σ_p -Tate term need not vanish in (2).

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