THE IMAGE OF THE SUSLIN–HUREWICZ MAP

ABSTRACT. For an infinite field F and an integer $n \geq 1$, Suslin constructed a natural homomorphism $h_n\colon K_n(F) \to K_n^M(F)$ from Quillen K-theory to Milnor K-theory whose composite with the canonical product map $K_n^M(F) \to K_n(F)$ is multiplication by $\pm(n-1)!$. We prove Suslin's conjecture that the image of h_n equals $(n-1)!\,K_n^M(F)$ for every $n \geq 1$ and every infinite field F. On the mth effective cover of algebraic K-theory for $m = \lfloor n/2 \rfloor + 1$, $\mathbf{P}^1$-stabilization turns the nth Chern class into a stable operation c_n^{st} over the prime field satisfying explicit Bott-shift recurrences, while Adams operations show that cover classes surject onto $K_n(F)/q$ away from the characteristic. Away from the prime 2 in characteristic zero, decomposing mod-q cover classes along the Bott tower into transfers of products of cyclotomic Bott elements and Milnor symbols determines the image of c_n^{st}. At the prime 2 in characteristic zero, the Bott-shift recurrences compare c_n^{st} with $\pm((n-1)!/2^r)\theta_r$ up to a 2-torsion operation whose vanishing reduces to odd n over $\mathbf{R}$, after which we evaluate θ_r over $F(\sqrt{-1})$ and descend to F.

1. INTRODUCTION

Let F be an infinite field. Homology stability for the general linear groups $GL_n(F)$ gives Suslin's natural homomorphism

$$h_n\colon K_n(F) \longrightarrow H_n(GL(F)) \cong H_n(GL_n(F)) \longrightarrow H_n(GL_n(F))/H_n(GL_{n-1}(F)) \cong K_n^M(F)$$

from Quillen K-theory to Milnor K-theory [18, §4]. Let $\iota_n\colon K_n^M(F) \to K_n(F)$ denote the canonical product map. Suslin proved that $h_n \circ \iota_n = \pm(n-1)!$ [18, Corollary 4.4], so $\mathrm{im}(h_n)$ contains $(n-1)!\,K_n^M(F)$, and he conjectured that equality holds for all $n \geq 1$.

For $n \leq 2$ we have $(n-1)! = 1$, so equality holds trivially. For infinite fields of characteristic different from 2, Suslin reduced the case $n = 3$ to the degree-3 Milnor conjecture on quadratic forms [18, Proposition 4.5]. For fields of characteristic different from 2 and 3, Asok, Fasel, and Williams proved the conjecture for $n = 5$ [4, Theorem 3.19], and Röndigs proved it for $n = 4$ [16, Theorem 4.8], both via unstable $\mathbf{A}^1$-homotopy groups. We establish Suslin's conjecture in full generality.

Theorem A. *For every infinite field F and every integer $n \geq 1$,*

$$\mathrm{im}\big(h_n\colon K_n(F) \longrightarrow K_n^M(F)\big) = (n-1)!\,K_n^M(F).$$

We work in the motivic stable homotopy category $\mathrm{SH}(S)$, with motivic spheres $S^{a,b}$ and bigraded suspension $\Sigma^{a,b}$. Over a field F, we write $\pi_{a,b}E(F) = [S^{a,b}, E]_{\mathrm{SH}(F)}$ for $E \in \mathrm{SH}(F)$, so $\pi_{-a,-b}H\mathbf{Z}(F) = H^{a,b}(F,\mathbf{Z})$ and $H^{n,n}(F,\mathbf{Z}) = K_n^M(F)$. For $a \in \mathbf{Z}$, let $\Sigma^{2a,a}\mathrm{SH}(S)^{\mathrm{eff}} \subseteq \mathrm{SH}(S)$ be the a-effective localizing subcategory, generated under colimits and extensions by $\Sigma^{p,q}\Sigma^\infty_+ U$ for smooth S-schemes U and $q \geq a$. We write f_aE for the a-effective cover (the right adjoint to this inclusion composed with the inclusion) and $s_aE = \mathrm{cofib}(f_{a+1}E \to f_aE)$ for the ath slice. For an integer $q \geq 1$, put $E/q = \mathrm{cofib}(q\colon E \to E)$, and write $E_{(\ell)}$ and $E[1/e]$ for coefficient localizations; morphism groups between localized spectra are taken in the localized category. Because $S^{a,b}$ and the a-effective generators are compact, f_a and s_a commute with $(-)/q$ and localization telescopes, and $\pi_{a,b}(E_{(\ell)})(F) \cong \pi_{a,b}E(F)_{(\ell)}$ and $\pi_{a,b}(E[1/e])(F) \cong \pi_{a,b}E(F)[1/e]$. When q is a power of ℓ, the localization map $E \to E_{(\ell)}$ induces an equivalence $E/q \xrightarrow{\simeq} E_{(\ell)}/q$, so any morphism $\alpha\colon E_{(\ell)} \to E'_{(\ell)}$ in $\mathrm{SH}(k)_{(\ell)}$ induces

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a morphism $E/q \to E'/q$ in $\mathrm{SH}(k)$. For any field extension F/k, we use the same symbol α for its base change to $\mathrm{SH}(F)_{(\ell)}$ and for the induced maps on $\pi_{*,*}(E_{(\ell)})(F)$ and $\pi_{*,*}(E/q)(F)$.

For each integer $n \geq 3$, write $n = 2r + j$ with $r = \lceil n/2 \rceil - 1 \geq 1$ and $j \in \{1, 2\}$, and set

$$m = \lfloor n/2 \rfloor + 1 = r + j, \qquad r = n - m, \qquad j = 2m - n \in \{1, 2\}, \tag{1.1}$$

so that $2m - 2 \leq n < 2m$ and $(-j - 2r, -m - r) = (-n, -n)$. Over any perfect field F, the effective cover $\mathrm{kgl} = f_0\mathrm{KGL}$ of the algebraic K-theory spectrum KGL sits in the Bott cofiber sequence

$$\Sigma^{2,1}\mathrm{kgl} \xrightarrow{\beta} \mathrm{kgl} \xrightarrow{u} H\mathbf{Z} \longrightarrow \Sigma^{3,1}\mathrm{kgl}.$$

Moreover, $f_m\mathrm{KGL} \simeq \Sigma^{2m,m}\mathrm{kgl}$, so $\pi_{n,0}(f_m\mathrm{KGL})(F) = \pi_{n-2m,-m}\mathrm{kgl}(F) = \pi_{-j,-m}\mathrm{kgl}(F)$, and the canonical morphism $f_m\mathrm{KGL} \to \mathrm{KGL}$ induces a homomorphism

$$\pi_{n,0}(f_m\mathrm{KGL})(F) = \pi_{-j,-m}\mathrm{kgl}(F) \longrightarrow \pi_{n,0}\mathrm{KGL}(F) = K_n(F).$$

We call elements of $\pi_{n,0}(f_m\mathrm{KGL})(F)$ *cover classes*. Write $\beta^s \colon \Sigma^{2s,s}\mathrm{kgl} \to \mathrm{kgl}$ for the s-fold Bott map. For a motivic spectrum E and an integer $s \geq 0$, the *Bott value* of a morphism $\gamma \colon \mathrm{kgl} \to \Sigma^{2s,s}E$ is the composite

$$S^{2s,s} \longrightarrow \Sigma^{2s,s}\mathrm{kgl} \xrightarrow{\beta^s} \mathrm{kgl} \xrightarrow{\gamma} \Sigma^{2s,s}E,$$

regarded as an element of $[S^{2s,s}, \Sigma^{2s,s}E] \cong \pi_{0,0}E$. When $s \geq 1$, put $\beta^*\gamma = \Sigma^{-2,-1}(\gamma \circ \beta) \colon \mathrm{kgl} \to \Sigma^{2s-2,s-1}E$; it has the same Bott value as γ.

The starting point of the proof is that h_n agrees up to sign with the nth Chern class $c_{n,n} \colon K_n(F) \to K_n^M(F)$, as follows from Williams's computation of $H^{*,*}(GL_n, \mathbf{Z})$ [21] and the unstable description of h_n due to Asok, Fasel, and Williams [4]. On $BGL = \Omega^\infty f_1\mathrm{KGL}$, the universal class c_n is unstable. On the deeper cover $f_m\mathrm{KGL} \simeq \Sigma^{2m,m}\mathrm{kgl}$, however, the $\mathbf{P}^1$-stabilization theorem of Asok, Bachmann, Elmanto, and Hopkins [1, 2] shows that $\mathrm{fib}(\Sigma^\infty\Omega^\infty\Sigma^{2m,m}\mathrm{kgl}[1/e] \to \Sigma^{2m,m}\mathrm{kgl}[1/e])$ is $2m$-effective. Because $2m > n$, the n-slice $\Sigma^{2n,n}H\mathbf{Z}[1/e]$ receives no nonzero maps from $2m$-effective spectra, so $c_n|_{\Omega^\infty f_m\mathrm{KGL}[1/e]}$ deloops uniquely to a stable morphism $C_n = \Sigma^{2m,m}c_n^{\mathrm{st}}$, and looping along $\mathbf{P}^1$ expresses c_{2r+2}^{st} and $\beta^*(c_{2r+1}^{\mathrm{st}})$ as integer multiples of c_{2r+1}^{st} and c_{2r-1}^{st}. Conversely, because $2m - 2 \leq n$, the slices below weight m contribute motivic cohomology of nonpositive degree and positive weight in total degree n. Those groups are uniquely ℓ-divisible away from the characteristic, and Adams operations kill the outgoing boundary maps at each step below weight m. Thus $m = \lfloor n/2 \rfloor + 1$ is the unique integer satisfying the two sufficient numerical bounds $2m - 2 \leq n < 2m$ used for cover lifting and stabilization.

Theorem B. *Let $n \geq 3$ be an integer, let m, r, j be as in (1.1), and let k be a prime field of exponential characteristic e. There exists a morphism*

$$c_n^{\mathrm{st}} \colon \mathrm{kgl}[1/e] \longrightarrow \Sigma^{2r,r}H\mathbf{Z}[1/e]$$

in $\mathrm{SH}(k)[1/e]$ with the following properties.

1. *The Bott value of c_n^{st} in $\pi_{0,0}H\mathbf{Z}[1/e](k) = \mathbf{Z}[1/e]$ equals $\pm(n-1)!$, and $(\beta^*)^{r-1}(c_n^{\mathrm{st}}) = \pm\frac{(n-1)!}{2}\, c_3^{\mathrm{st}}$ in $[\mathrm{kgl}[1/e], \Sigma^{2,1}H\mathbf{Z}[1/e]]$.*
2. *For every infinite perfect field F over k and every cover class $y \in \pi_{-j,-m}\mathrm{kgl}(F)$ with image $\bar{y} \in K_n(F)$, the image $c_n^{\mathrm{st}}(y)$ of y in $\pi_{-j,-m}(\Sigma^{2r,r}H\mathbf{Z}[1/e])(F) = K_n^M(F)[1/e]$ satisfies*
$$h_n(\bar{y}) = \pm c_n^{\mathrm{st}}(y) \qquad \text{in } K_n^M(F)[1/e].$$
3. *For every perfect field F over k and every power q of a prime $\ell \neq e$, every element of $K_n(F)$ is congruent modulo q to the image of a cover class.*

Theorem B reduces Theorem A away from the characteristic to computing the image of c_n^{st} on finite-coefficient cover classes. We decompose $\pi_{-j,-m}(\mathrm{kgl}/q)(F)$ along the Bott tower using Kahn's cyclotomic corestriction theorem [10]: away from $\ell = 2$ over $\mathbf{Q}$ (and for sufficiently large 2-powers over $\mathbf{F}_p$), every class in $H^a_{\mathrm{et}}(F, \mu_q^{\otimes(a+s)})$ $(a, s \geq 1)$ is corestricted from the minimal extension Fk_s/F splitting $\mu_q^{\otimes s}$. Lifting generators of $H^0_{\mathrm{et}}(k_s, \mu_q^{\otimes s})$

to Bott elements $\mathfrak{b}_s \in \pi_{0,-s}(\mathrm{kgl}/q)(k_s)$ expresses every mod-q cover class in terms of $\beta^{r-s}\,\mathrm{Tr}_{Fk_s/F}(\mathfrak{b}_s \cdot \widehat{A}_s)$ $(1 \le s \le r)$ and $\beta^r x_0$ with $x_0 \in \pi_{-n,-n}(\mathrm{kgl}/q)(F) \cong K^M_n(F)/q$. Because c^{st}_n is defined over k, the value of $(\beta^*)^{r-s}(c^{\mathrm{st}}_n)$ on $\mathfrak{b}_s$ lies in $K^M_{2s}(k_s)/q$, which vanishes by cohomological dimension except when $k = \mathbf{Q}$ and $s = 1$; there $(\beta^*)^{r-1}(c^{\mathrm{st}}_n) = \pm\frac{(n-1)!}{2}\,c^{\mathrm{st}}_3$ and 2 is invertible at odd primes ℓ.

Theorem C. *Let $n \ge 3$ be an integer, let m, r, j be as in (1.1), let ℓ be a prime, let k be either $\mathbf{Q}$ with ℓ odd or $\mathbf{F}_p$ with $\ell \ne p$, and let $q = \ell^\nu$ with $\nu \ge 1$. When $\ell = 2$, assume also that $\nu \ge 3$, and that $\nu > v_2(p+1)$ if $p \equiv 3 \pmod 4$. For every perfect field F over k, the homomorphism induced by c^{st}_n of Theorem B,*

$$c^{\mathrm{st}}_n \colon \pi_{-j,-m}(\mathrm{kgl}/q)(F) \longrightarrow \pi_{-j,-m}(\Sigma^{2r,r}H\mathbf{Z}/q)(F) = K^M_n(F)/q,$$

has image $(n-1)!\,K^M_n(F)/q$.

At $\ell = 2$ in characteristic zero, real places make $K^M_{2s}(k_s)/2$ nonzero, obstruct Kahn's corestriction over F, and allow 2-torsion in $[\mathrm{kgl}_{(2)}, \Sigma^{2r,r}H\mathbf{Z}_{(2)}]$ over $\mathbf{Q}$. Using the Steenrod algebra over $\mathbf{Q}$ and the Bott-shift recurrences $c^{\mathrm{st}}_{2r+2} = \pm(2r+1)\,c^{\mathrm{st}}_{2r+1}$ and $\beta^*(c^{\mathrm{st}}_{2r+1}) = \pm 2r(2r-1)\,c^{\mathrm{st}}_{2r-1}$, we construct operations $\theta_s \colon \mathrm{kgl}_{(2)} \to \Sigma^{2s,s}H\mathbf{Z}_{(2)}$ of Bott value 2^s satisfying $\beta^*\theta_s = 2\theta_{s-1}$ and show for $r \ge 2$ that the 2-localization of c^{st}_n $(n \in \{2r+1, 2r+2\})$ equals $\pm((n-1)!/2^r)\theta_r + \kappa^{(r)}$ for a single 2-torsion operation $\kappa^{(r)}$ factoring through $H\mathbf{Z}_{(2)}$. Evaluating $\kappa^{(r)}$ on a real cover class for odd $n = 2r+1$ via $Sq^2(\bar{c}_{2r})$ proves that $\kappa^{(r)}$ vanishes on all cover classes. Over $L = F(\sqrt{-1})$, Kahn's decomposition of $K_n(L; \mathbf{Z}/q)$ [12] identifies θ_r with 2^r times the projection onto $K^M_n(L)/q$, and quadratic descent gives $\theta_r(y) \in 2^r A + A[2]$, where $A = K^M_n(F)_{(2)}$. For $n \ge 5$, the integer $(n-1)!/2^r$ is even, so it annihilates the remaining absolute 2-torsion and yields the required factorial divisibility.

Theorem D. *Let $n \ge 3$ be an integer and put $m = \lfloor n/2 \rfloor + 1$. For every field F of characteristic zero and every cover class $y \in \pi_{n,0}(f_m\mathrm{KGL})(F)$, the image of $c^{\mathrm{st}}_n(y) \in K^M_n(F)$ in $K^M_n(F)_{(2)}$ lies in $2^{v_2((n-1)!)}\,K^M_n(F)_{(2)}$.*

Proof of Theorem A assuming Theorems B to D. Since $h_n \circ \iota_n = \pm(n-1)!$, the image of h_n contains $(n-1)!\,K^M_n(F)$, which equals $K^M_n(F)$ for $n \le 2$. Fix $n \ge 3$ and $x \in K_n(F)$. For coprime integers c, c' and any abelian group M, Bézout's identity $sc + tc' = 1$ gives $cM \cap c'M = cc'M$, so $(n-1)!M = \bigcap_\ell \ell^{v_\ell((n-1)!)}M$ over all primes ℓ. Moreover, since $M/\ell^a M \to M_{(\ell)}/\ell^a M_{(\ell)}$ is an isomorphism for every $a \ge 0$, an element of M lies in $\ell^a M$ if and only if its image in $M_{(\ell)}$ lies in $\ell^a M_{(\ell)}$. Set $v_\ell = v_\ell((n-1)!)$; it suffices to show for each prime ℓ that $h_n(x)$ maps into $\ell^{v_\ell}K^M_n(F)_{(\ell)}$.

(1) Suppose $\mathrm{char}(F) = 0$ and $\ell = 2$. Put $q = 2^{v_2}$. By Theorem B(3) and (2) (with $k = \mathbf{Q}$ and $e = 1$), there exists $y \in \pi_{n,0}(f_m\mathrm{KGL})(F)$ whose image $\bar{y} \in K_n(F)$ satisfies $x \equiv \bar{y} \pmod q$ and $h_n(\bar{y}) = \pm c^{\mathrm{st}}_n(y)$ in $K^M_n(F)$. Then $h_n(x) - h_n(\bar{y}) \in qK^M_n(F)$, while Theorem D places the image of $c^{\mathrm{st}}_n(y)$ in $2^{v_2}K^M_n(F)_{(2)}$. Hence $h_n(x)$ maps into $2^{v_2}K^M_n(F)_{(2)}$.

(2) Suppose F is perfect of exponential characteristic e, $\ell \ne e$, and either ℓ is odd or $\mathrm{char}(F) = p > 0$. Choose $q = \ell^\nu$ with $\nu \ge v_\ell$ satisfying the hypotheses of Theorem C over the prime field k of F. By Theorem B(3) and (2), there exists $y \in \pi_{n,0}(f_m\mathrm{KGL})(F)$ with image $\bar{y} \in K_n(F)$ such that $x \equiv \bar{y} \pmod q$ and $h_n(\bar{y}) = \pm c^{\mathrm{st}}_n(y)$ in $K^M_n(F)[1/e]$. Applying Theorem C to the image of y in $\pi_{-j,-m}(\mathrm{kgl}/q)(F)$ shows that $c^{\mathrm{st}}_n(y) \bmod q$ lies in $(n-1)!\,K^M_n(F)/q$. In $K^M_n(F)_{(\ell)}$, we therefore have

$$c^{\mathrm{st}}_n(y) \in (n-1)!\,K^M_n(F)_{(\ell)} + qK^M_n(F)_{(\ell)} = \ell^{v_\ell}K^M_n(F)_{(\ell)},$$

so $h_n(x)$ maps into $\ell^{v_\ell}K^M_n(F)_{(\ell)}$.

(3) Suppose F is imperfect of characteristic $p > 0$ and $\ell \ne p$. Its perfect closure F^{perf} is an infinite perfect field, so case (2) and naturality of h_n imply that $\mathrm{res}_{F^{\mathrm{perf}}/F}(h_n(x))$

lies in $\ell^{v_\ell} K_n^M(F^{\mathrm{perf}})$. Since $K_n^M(F^{\mathrm{perf}})$ is the filtered colimit of $K_n^M(F')$ over finite purely inseparable subextensions F'/F, there exist such an extension F'/F of degree p^d and $w \in K_n^M(F')$ with $\mathrm{res}_{F'/F}(h_n(x)) = \ell^{v_\ell} w$. Applying the norm $N_{F'/F}\colon K_n^M(F') \to K_n^M(F)$ gives

$$p^d h_n(x) = N_{F'/F}(\mathrm{res}_{F'/F}(h_n(x))) = \ell^{v_\ell} N_{F'/F}(w) \qquad \text{in } K_n^M(F).$$

Choosing $b \in \mathbf{Z}$ with $bp^d \equiv 1 \pmod{\ell^{v_\ell}}$ gives $h_n(x) \in \ell^{v_\ell} K_n^M(F)$.

(4) Suppose $\ell = p = \mathrm{char}(F) > 0$. By Geisser–Levine [5, Theorem 8.1], the product map ι_n induces an isomorphism $K_n^M(F)/p^{v_p} \xrightarrow{\sim} K_n(F)/p^{v_p}$. Writing $x = \iota_n(b) + p^{v_p} z$ with $b \in K_n^M(F)$ and $z \in K_n(F)$ gives

$$h_n(x) = \pm(n-1)!\, b + p^{v_p} h_n(z) \in p^{v_p} K_n^M(F). \qquad \square$$

In Section 2 we study the effective covers $f_a\mathrm{KGL}$, prove Theorem B(3), and record the action of sphere lifts and transfers on mod-q cover classes. In Section 3 we compare h_n with $c_{n,n}$, construct c_n^{st}, and establish its Bott-shift recurrences to finish the proof of Theorem B. In Section 4 we prove the Bott tower decomposition and Theorem C. In Section 5 we construct θ_s over $\mathbf{Q}$, deduce Theorem D for $n = 3, 4$, and show that the 2-torsion summand $\kappa^{(r)}$ of c_n^{st} vanishes on cover classes for $n \geq 5$. In Section 6 we evaluate θ_r over fields containing $\sqrt{-1}$ and descend to complete the proof of Theorem D.

2. Effective covers of algebraic K-theory

2.1. **Slices, connectivity, and base change.** Let F be a perfect field. Over F, the effective cover $\mathrm{kgl} = f_0\mathrm{KGL}$ sits in the Bott cofiber sequence

$$\Sigma^{2,1}\mathrm{kgl} \xrightarrow{\beta} \mathrm{kgl} \xrightarrow{u} H\mathbf{Z} \xrightarrow{\delta_0} \Sigma^{3,1}\mathrm{kgl}, \tag{2.1}$$

and $f_a\mathrm{KGL} \simeq \Sigma^{2a,a}\mathrm{kgl}$ for every $a \geq 0$, with $f_{a+1}\mathrm{KGL} \to f_a\mathrm{KGL}$ corresponding to $\Sigma^{2a,a}\beta$ and $s_a\mathrm{KGL} \simeq \Sigma^{2a,a} H\mathbf{Z}$ [1, §7.3], [2, §2.2]. Under $f_a\mathrm{KGL} \simeq \Sigma^{2a,a}\mathrm{kgl}$, the canonical map $f_a\mathrm{KGL} \to \mathrm{KGL}$ is the composite of $\beta^a\colon \Sigma^{2a,a}\mathrm{kgl} \to \mathrm{kgl}$ with the counit $\mathrm{kgl} \to \mathrm{KGL}$.

Lemma 2.1. *Let k be a perfect field of exponential characteristic e, let $\ell \neq e$ be a prime, and let Λ be $\mathbf{Z}[1/e]$, $\mathbf{Z}_{(\ell)}$, or $\mathbf{Z}/\ell$. In $\mathrm{SH}(k)$ (localized at $1/e$ when $\Lambda = \mathbf{Z}[1/e]$, or at ℓ when $\Lambda = \mathbf{Z}_{(\ell)}$ or $\mathbf{Z}/\ell$), precomposition with $u\colon \mathrm{kgl} \to H\mathbf{Z}$ and with the unit $\mathbf{1} \to H\mathbf{Z}$ induces isomorphisms*

$$[\mathrm{kgl}, \Sigma^{a,0} H\Lambda] \xleftarrow[\sim]{u^*} [H\mathbf{Z}, \Sigma^{a,0} H\Lambda] \xrightarrow{\sim} H^{a,0}(k, \Lambda)$$

for every $a \in \mathbf{Z}$. In particular, $[\mathrm{kgl}, H\Lambda] = \Lambda \cdot u$ and $[\mathrm{kgl}, \Sigma^{a,0} H\Lambda] = 0$ for all $a \neq 0$.

Proof. Since $s_0\mathbf{1} \simeq H\mathbf{Z}$ and s_0 commutes with localization and $(-)/\ell$, the spectrum $\Sigma^{a,0} H\Lambda$ is a 0-slice. In both (2.1) and $f_1\mathbf{1} \to \mathbf{1} \to H\mathbf{Z} \to \Sigma^{1,0} f_1\mathbf{1}$, the outer terms are 1-effective, so maps from their localizations into $\Sigma^{a,0} H\Lambda$ vanish. Both u^* and restriction along the unit are therefore isomorphisms. Finally, $H^{a,0}(k, \Lambda)$ equals Λ for $a = 0$ and 0 for $a \neq 0$ [14, Theorem 3.6 and Corollary 4.2]. $\square$

Because kgl is very effective, it is connective in Morel's homotopy t-structure [8, Corollary 5.2]: its homotopy modules vanish in simplicial degrees $a - b < 0$, so $\pi_{a,b}\mathrm{kgl}(F) = 0$ whenever $a < b$. Connective spectra are closed under cofibers, so $\pi_{a,b}(\mathrm{kgl}/q)(F) = 0$ for $a < b$ and every $q \geq 1$. Taking $(a, b) = (v - 2w, -w)$ gives, for all $v < w$,

$$\pi_{v,0}(f_w\mathrm{KGL})(F) = \pi_{v-2w,-w}\mathrm{kgl}(F) = 0, \qquad \pi_{v,0}(f_w\mathrm{KGL}/q)(F) = 0. \tag{2.2}$$

Note also that $\pi_{v,0}(s_w\mathrm{KGL})(F) = H^{2w-v,w}(F, \mathbf{Z})$ vanishes whenever $2w - v > w$ [14, Theorem 3.6], as well as when $w = 0$ and $v \neq 0$.

Lemma 2.2. *For every extension of perfect fields $g\colon \mathrm{Spec}\, F' \to \mathrm{Spec}\, F$, the pullback functor g^* commutes up to canonical equivalence with f_a ($a \in \mathbf{Z}$) and with Ω^∞.*

Proof. When g is smooth, its left adjoint $g_\sharp$ preserves a-effective spectra and commutes with Σ^∞, so g^* commutes with the right adjoints f_a and Ω^∞. In general, g is essentially smooth [9, Lemma 2.10], meaning that $\operatorname{Spec} F'$ is the limit of a cofiltered diagram of smooth affine F-schemes $\operatorname{Spec} A_i$. For any $E \in \mathrm{SH}(F)$, the natural map $g^* f_a E \to f_a g^* E$ in $\mathrm{SH}(F')$ is an equivalence if and only if it induces an isomorphism on $[\Sigma^{p,q}\Sigma^\infty_+ V', -]_{\mathrm{SH}(F')}$ for every smooth F'-scheme V' and every $q \geq a$. Any such V' is pulled back from a smooth A_i-scheme V_i at some stage i, where the comparison map reduces to the smooth case over A_i and the continuity isomorphism for SH [9, §2.2 and Lemma 2.8]. Testing Ω^∞ on smooth F'-schemes reduces to the smooth case by the same continuity property. □

2.2. **Lifting to cover classes.**

Lemma 2.3. *Let F be an arbitrary field, let ℓ be a prime different from $\mathrm{char}(F)$, and let $w > 0$ and $a \leq 0$. Then $H^{a,w}(F, \mathbf{Z}_{(\ell)})$ is uniquely ℓ-divisible.*

Proof. By the Beilinson–Lichtenbaum comparison theorem [20, Theorem 6.17],

$$H^{b,w}(F, \mathbf{Z}/\ell^s) \cong H^b_{\mathrm{et}}(F, \mu_{\ell^s}^{\otimes w}) \qquad \text{for all } b \leq w \text{ and } s \geq 1.$$

Thus $H^{b,w}(F, \mathbf{Z}/\ell) = 0$ for all $b < 0$, and the long exact sequence associated to $\mathbf{Z}_{(\ell)} \xrightarrow{\ell} \mathbf{Z}_{(\ell)} \to \mathbf{Z}/\ell$ shows that multiplication by ℓ on $H^{a,w}(F, \mathbf{Z}_{(\ell)})$ is bijective for $a < 0$ and injective for $a = 0$.

Now let $x \in H^{0,w}(F, \mathbf{Z}_{(\ell)})$. By continuity of motivic cohomology, x is the restriction of some $x_0 \in H^{0,w}(F_0, \mathbf{Z}_{(\ell)})$ over a subfield $F_0 \subseteq F$ finitely generated over the prime field. Because inverse limits are left exact, the reductions of x_0 modulo ℓ^s for $s \geq 1$ form a compatible system in

$$\varprojlim_s H^0_{\mathrm{et}}(F_0, \mu_{\ell^s}^{\otimes w}) = \Big(\varprojlim_s \mu_{\ell^s}^{\otimes w}\Big)^{\mathrm{Gal}(F_0^{\mathrm{sep}}/F_0)} = \mathbf{Z}_\ell(w)^{\mathrm{Gal}(F_0^{\mathrm{sep}}/F_0)}.$$

If $\mathrm{char}(F_0) = 0$, the algebraic closure of $\mathbf{Q}$ in F_0 is a number field, so the ℓ-adic cyclotomic character of F_0 and its wth power have infinite image. If $\mathrm{char}(F_0) = p > 0$, the constant field of F_0 is finite and geometric Frobenius acts on $\mathbf{Z}_\ell(w)$ by an integer power of $p^w \neq 1$. In either case, some element of $\mathrm{Gal}(F_0^{\mathrm{sep}}/F_0)$ acts on the free rank-one $\mathbf{Z}_\ell$-module $\mathbf{Z}_\ell(w)$ by a scalar different from 1, so the Galois invariants vanish. Projecting the inverse limit to $s = 1$ shows that x_0 maps to 0 in $H^{0,w}(F_0, \mathbf{Z}/\ell)$, hence lies in $\ell H^{0,w}(F_0, \mathbf{Z}_{(\ell)})$, so its restriction x lies in $\ell H^{0,w}(F, \mathbf{Z}_{(\ell)})$. □

Let F be a perfect field and let $k \geq 2$ be an integer. In the motivic Snaith model $\mathrm{KGL} \simeq \Sigma^\infty_+ \mathbf{P}^\infty[\tilde{\beta}^{-1}]$ [6, §1.1 and Theorem 4.17], the kth tensor power endomorphism of $\mathbf{P}^\infty \simeq BGL_1$ sends the Bott class $[\mathcal{O}(1)] - 1 \in \widetilde{K}_0(\mathbf{P}^1)$ to $[\mathcal{O}(k)] - 1 = k([\mathcal{O}(1)] - 1)$, since $([\mathcal{O}(1)] - 1)^2 = 0$. By the universal property of Bott inversion, this induces a ring map $\Psi^k \colon \mathrm{KGL}[1/k] \to \mathrm{KGL}[1/k]$ sending $\beta \in \pi_{2,1}\mathrm{KGL}$ to $k\beta$. Functoriality of f_w and s_w gives compatible endomorphisms of the slice tower of $\mathrm{KGL}[1/k]$. Because the unit induces $s_0\mathbf{1} \xrightarrow{\simeq} s_0\mathrm{KGL}$, the map $s_0\Psi^k$ is the identity on $H\mathbf{Z}[1/k]$, and since Ψ^k is a ring map with $\Psi^k(\beta) = k\beta$, the endomorphism $s_w\Psi^k$ acts as multiplication by k^w on $s_w\mathrm{KGL}[1/k] \simeq \Sigma^{2w,w} H\mathbf{Z}[1/k]$.

Proposition 2.4. *Let F be a perfect field, let $n \geq 3$ be an integer, and put $m = \lfloor n/2 \rfloor + 1$. For every prime $\ell \neq \mathrm{char}(F)$ and every power q of ℓ, every element of $K_n(F)$ is congruent modulo q to the image of an element of $\pi_{n,0}(f_m\mathrm{KGL})(F)$.*

Proof. Choose an integer $k \geq 2$ coprime to ℓ, so Ψ^k acts on the slice tower of $\mathrm{KGL}_{(\ell)}$. For $0 \leq w \leq n+1$, define $M_w = \mathrm{cofib}(f_{n+1}\mathrm{KGL}_{(\ell)} \to f_w\mathrm{KGL}_{(\ell)})$. By (2.2), $\pi_{v,0}(f_{n+1}\mathrm{KGL}_{(\ell)})(F) = 0$ for $v \leq n$, which identifies $\pi_{n,0}M_m(F)$ with $\pi_{n,0}(f_m\mathrm{KGL})(F)_{(\ell)}$ and $\pi_{n,0}M_0(F)$ with $K_n(F)_{(\ell)}$. Since $A/q \to A_{(\ell)}/q$ is an isomorphism for every abelian group A, it suffices to show for each $0 \leq w < m$ that $\pi_{n,0}M_{w+1}(F) \to \pi_{n,0}M_w(F)$ is surjective modulo q.

Fix $0 \le w < m$. Applying the octahedral axiom to $f_{n+1}\mathrm{KGL}_{(\ell)} \to f_{w+1}\mathrm{KGL}_{(\ell)} \to f_w\mathrm{KGL}_{(\ell)}$ gives a Ψ^k-equivariant cofiber sequence $M_{w+1} \to M_w \to \Sigma^{2w,w}H\mathbf{Z}_{(\ell)}$ and hence an exact sequence

$$\pi_{n,0}M_{w+1}(F) \xrightarrow{j_w} \pi_{n,0}M_w(F) \longrightarrow H^{2w-n,w}(F,\mathbf{Z}_{(\ell)}) \xrightarrow{\partial_w} \pi_{n-1,0}M_{w+1}(F).$$

For $w = 0$, we have $H^{-n,0}(F,\mathbf{Z}_{(\ell)}) = 0$ since $n \ge 3$, so j_0 is surjective. Suppose $1 \le w < m$. Then $2w - n \le 2m - 2 - n \le 0$, so $D_w = H^{2w-n,w}(F,\mathbf{Z}_{(\ell)})$ is uniquely ℓ-divisible by Lemma 2.3. Because M_{w+1} is an iterated extension of $\Sigma^{2i,i}H\mathbf{Z}_{(\ell)}$ for $w+1 \le i \le n$, the operator $P_{>w}(\Psi^k) = \prod_{i=w+1}^{n}(\Psi^k - k^i)$ annihilates $\pi_{n-1,0}M_{w+1}(F)$, whereas on $\Sigma^{2w,w}H\mathbf{Z}_{(\ell)}$ it acts as multiplication by the nonzero integer $N_w = \prod_{i=w+1}^{n}(k^w - k^i)$. Equivariance of ∂_w gives $N_w\partial_w = P_{>w}(\Psi^k)\partial_w = 0$. Since D_w is ℓ-divisible, its quotient $\mathrm{im}(\partial_w)$ is both ℓ-divisible and killed by $\ell^{v_\ell(N_w)}$, so $\partial_w = 0$. Thus $0 \to \mathrm{im}(j_w) \to \pi_{n,0}M_w(F) \to D_w \to 0$ is exact; because multiplication by q is bijective on D_w, the snake lemma gives $\mathrm{im}(j_w)/q\,\mathrm{im}(j_w) \xrightarrow{\sim} \pi_{n,0}M_w(F)/q$, so j_w is surjective modulo q. □

2.3. **Finite-coefficient covers.** Let F be a perfect field, let $\ell \ne \mathrm{char}(F)$ be a prime, and let q be a power of ℓ. By (2.1), the kernel of the edge homomorphism

$$u\colon \pi_{-a,-b}(\mathrm{kgl}/q)(F) \longrightarrow H^{a,b}(F,\mathbf{Z}/q)$$

equals the image of $\beta\colon \pi_{-a-2,-b-1}(\mathrm{kgl}/q)(F) \to \pi_{-a,-b}(\mathrm{kgl}/q)(F)$.

Lemma 2.5. *Let F be a perfect field, let $\ell \ne \mathrm{char}(F)$ be a prime, and let q be a power of ℓ. For every $a \ge 0$, the counit $f_0\mathrm{KGL} \to \mathrm{KGL}$ identifies $\pi_{a,0}(\mathrm{kgl}/q)(F)$ with $K_a(F;\mathbf{Z}/q)$. For every $s \ge 0$, the canonical map*

$$\pi_{2s,0}(f_s\mathrm{KGL}/q)(F) \longrightarrow K_{2s}(F;\mathbf{Z}/q)$$

is an isomorphism.

Proof. The adjunction for f_0 gives $\pi_{v,0}\mathrm{kgl}(F) \xrightarrow{\sim} K_v(F)$ for $v \ge 0$, while $\pi_{v,0}\mathrm{kgl}(F) = \pi_{v,0}\mathrm{KGL}(F) = 0$ for $v < 0$ by (2.2) and regularity of F. Applying the five-lemma to the universal coefficient sequences for kgl/q and KGL/q gives $\pi_{a,0}(\mathrm{kgl}/q)(F) \xrightarrow{\sim} K_a(F;\mathbf{Z}/q)$ for all $a \ge 0$. For the second statement, $\mathrm{cofib}(f_s\mathrm{KGL}/q \to \mathrm{kgl}/q)$ is an iterated extension of $\Sigma^{2w,w}H\mathbf{Z}/q$ for $0 \le w < s$, whose homotopy groups in degrees $2s$ and $2s+1$ are $H^{2w-2s,w}(F,\mathbf{Z}/q) = 0$ and $H^{2w-2s-1,w}(F,\mathbf{Z}/q) = 0$ since $2w - 2s < 0$. The long exact sequence therefore gives $\pi_{2s,0}(f_s\mathrm{KGL}/q)(F) \xrightarrow{\sim} \pi_{2s,0}(\mathrm{kgl}/q)(F) \cong K_{2s}(F;\mathbf{Z}/q)$. □

Lemma 2.6. *Let F be a perfect field, let $\ell \ne \mathrm{char}(F)$ be a prime, let q be a power of ℓ, and let $s \ge 0$. If $\mathrm{cd}_\ell(F) \le 1$, then the edge map*

$$u\colon \pi_{2s,0}(f_s\mathrm{KGL}/q)(F) = \pi_{0,-s}(\mathrm{kgl}/q)(F) \longrightarrow H^{0,s}(F,\mathbf{Z}/q) \tag{2.3}$$

is an isomorphism.

Proof. The fiber of $f_s\mathrm{KGL}/q \to s_s\mathrm{KGL}/q$ is $f_{s+1}\mathrm{KGL}/q$, so it suffices to check that $\pi_{v,0}(f_{s+1}\mathrm{KGL}/q)(F) = 0$ for $v \in \{2s-1, 2s\}$. Fix such a v and let $w > s$. If $2w - v \le w$, then $2w - v \ge 2 > \mathrm{cd}_\ell(F)$, so

$$\pi_{v,0}(s_w\mathrm{KGL}/q)(F) = H^{2w-v,w}(F,\mathbf{Z}/q) \cong H^{2w-v}_{\mathrm{et}}(F,\mu_q^{\otimes w}) = 0.$$

If $2w - v > w$, then $H^{2w-v,w}(F,\mathbf{Z}/q) = 0$ because the degree exceeds the weight. Hence $\pi_{v,0}(f_{w+1}\mathrm{KGL}/q)(F) \to \pi_{v,0}(f_w\mathrm{KGL}/q)(F)$ is surjective for every $w > s$. Since the source vanishes for $w \ge v$ by (2.2), downward induction gives $\pi_{v,0}(f_{s+1}\mathrm{KGL}/q)(F) = 0$. □

For every perfect field F and every $a \ge 0$, both $\pi_{-a-2,-a-1}$ and $\pi_{-a-3,-a-1}$ vanish on kgl and on kgl/q by connectivity, so (2.1) induces edge isomorphisms

$$u\colon \pi_{-a,-a}\mathrm{kgl}(F) \xrightarrow{\sim} K^M_a(F), \qquad u\colon \pi_{-a,-a}(\mathrm{kgl}/q)(F) \xrightarrow{\sim} K^M_a(F)/q. \tag{2.4}$$

2.4. **Sphere lifts and transfers.** Let F be a perfect field. Every morphism in $\mathrm{SH}(F)$ is linear over $\pi_{*,*}\mathbf{1}(F)$ rather than over $\pi_{*,*}\mathrm{kgl}(F)$; to pass Milnor symbols across arbitrary stable operations out of kgl, we lift them to $\pi_{*,*}\mathbf{1}(F)$. A unit $v \in F^\times$ defines a pointed map $\operatorname{Spec} F_+ \to \mathbf{G}_m \simeq S^{1,1}$ and thus a class $[v] \in \pi_{-1,-1}\mathbf{1}(F)$, whose image under $\mathbf{1} \to H\mathbf{Z}$ is $\{v\} \in H^{1,1}(F,\mathbf{Z})$ [14, Theorem 4.1]. Given $A \in K^M_a(F)$ and a presentation $A = \sum_i \{v_{i,1},\dots,v_{i,a}\}$, we call $\widehat{A} = \sum_i [v_{i,1}]\cdots[v_{i,a}] \in \pi_{-a,-a}\mathbf{1}(F)$ a *sphere lift* of A. On $H\mathbf{Z}$ (and $H\mathbf{Z}/q$), multiplication by $\widehat{A}$ is multiplication by A (and by its mod-q reduction $\bar{A}$), while under $\mathbf{1} \to \mathrm{kgl}$ the class $\widehat{A}$ maps to $u^{-1}(A)$ along (2.4).

Lemma 2.7. *Let F be a perfect field. For every $v \in F^\times$, the composite*

$$\pi_{-1,-1}\mathbf{1}(F) \longrightarrow \pi_{-1,-1}\mathrm{kgl}(F) \longrightarrow \pi_{-1,-1}\mathrm{KGL}(F) \xrightarrow[\sim]{\cdot\beta} \pi_{1,0}\mathrm{KGL}(F) = K_1(F)$$

sends $[v]$ to $\iota_1(v)$. Consequently, under Bott periodicity, any sphere lift $\widehat{A}$ of $A \in K^M_a(F)$ acts on $K_*(F)$ and $K_*(F;\mathbf{Z}/q)$ as multiplication by $\iota_a(A)$.

Proof. Under $f_1\mathrm{KGL} \simeq \Sigma^{2,1}\mathrm{kgl}$, the map $\pi_{1,0}(f_1\mathrm{KGL})(F) \to K_1(F)$ agrees on $\pi_{-1,-1}\mathrm{kgl}(F)$ with multiplication by β after $\pi_{-1,-1}\mathrm{kgl}(F) \to \pi_{-1,-1}\mathrm{KGL}(F)$. On $BGL = \Omega^\infty f_1\mathrm{KGL}$, the map $\Omega^\infty(\Sigma^{2,1}u)$ represents a class in $H^{2,1}(BGL,\mathbf{Z}) = \mathbf{Z}\cdot c_1$ [7, §6.1]. On the Bott generator $[\mathcal{O}(1)] - 1 \in \widetilde{K}_0(\mathbf{P}^1)$, represented by the unit $S^{2,1} \to \Sigma^{2,1}\mathrm{kgl}$, both $\Omega^\infty(\Sigma^{2,1}u)$ and c_1 take the value $1 \in \widetilde{H}^{2,1}(\mathbf{P}^1,\mathbf{Z}) \cong \mathbf{Z}$, so $\Omega^\infty(\Sigma^{2,1}u) = c_1$. This gives a commutative diagram

$$\begin{array}{ccccc}
\pi_{1,0}(f_1\mathrm{KGL})(F) & \longrightarrow & \pi_1 BGL(F) = K_1(F) & \xrightarrow[\sim]{\det} & \pi_1 BGL_1(F) = F^\times \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle c_{1,1}} & \swarrow{\scriptstyle \mathrm{id}} & \\
H^{1,1}(F,\mathbf{Z}) & \xrightarrow[=]{} & H^{1,1}(F,\mathbf{Z}) = F^\times, & &
\end{array}$$

where the right-hand triangle commutes because c_1 is pulled back from BGL_1 along det and Ωc_1 on $GL_1 = \mathbf{G}_m$ is the coordinate class. Evaluating along the image of $[v]$ shows that its image in $K_1(F)$ has determinant $u([v]) = \{v\} = v$, hence equals $\iota_1(v)$. Since $\mathbf{1} \to \mathrm{KGL}$ is a ring map and $\mathrm{KGL}/q = \mathrm{KGL} \wedge \mathbf{1}/q$ is a KGL-module, any sphere lift $\widehat{A}$ acts on both $\pi_{*,*}\mathrm{KGL}(F)$ and $\pi_{*,*}(\mathrm{KGL}/q)(F)$ through its image $\beta^{-a}\iota_a(A) \in \pi_{-a,-a}\mathrm{KGL}(F)$. □

For a finite separable extension L/F, precomposition with the finite étale transfer morphism $\Sigma^\infty_+ \operatorname{Spec} F \to \Sigma^\infty_+ \operatorname{Spec} L$ in $\mathrm{SH}(F)$ defines a homomorphism

$$\mathrm{Tr}_{L/F}\colon \pi_{*,*}E(L) \longrightarrow \pi_{*,*}E(F)$$

that commutes with every morphism in $\mathrm{SH}(F)$. On KGL and KGL/q it agrees with the pushforward on algebraic K-theory, and on $H\mathbf{Z}$ and $H\mathbf{Z}/q$ it agrees with corestriction on motivic and étale cohomology [8, Remark 5.8 and Proposition 6.2].

3. The Chern class as a stable operation

3.1. **Comparison with Suslin's map.** For $n \geq 1$, a pointed map $BGL \to \Omega^\infty\Sigma^{2n,n}H\mathbf{Z}$ representing the universal Chern class $c_n \in H^{2n,n}(BGL,\mathbf{Z})$ induces on $\pi_n BGL(F) = K_n(F)$ a homomorphism

$$c_{n,n}\colon K_n(F) \longrightarrow H^{n,n}(F,\mathbf{Z}) = K^M_n(F).$$

Let $J_{n-1} \to \mathbf{P}^{n-1}$ be the affine Jouanolou torsor parametrizing direct-sum decompositions $\mathcal{O}^n = Q \oplus L$ with $\mathrm{rk}(L) = 1$, so $J_{n-1} \to \mathbf{P}^{n-1}$ is an $\mathbf{A}^1$-equivalence, and let

$$w_n\colon \mathbf{G}_m \wedge (J_{n-1})_+ \longrightarrow GL_n$$

be the morphism sending $(\lambda,(Q,L))$ to $\mathrm{id}_Q \oplus \lambda\cdot\mathrm{id}_L$ [21, Proposition 16]. Write $\sigma \in \widetilde{H}^{1,1}(\mathbf{G}_m,\mathbf{Z})$ for the coordinate class and $x = c_1(L) \in H^{2,1}(\mathbf{P}^{n-1},\mathbf{Z}) \cong H^{2,1}(J_{n-1},\mathbf{Z})$.

Theorem 3.1 (Williams)**.** *Let F be a field and $n \geq 2$.*

(1) For every $i \geq 2$, pullback along w_n is an isomorphism

$$w_n^*\colon H^{2i-1,i}(GL_n,\mathbf{Z}) \xrightarrow{\sim} \widetilde{H}^{2i-1,i}(\mathbf{G}_m \wedge \mathbf{P}^{n-1}_+,\mathbf{Z}).$$

(2) *Let* $p_1\colon GL_n \to \mathbf{A}^n \setminus 0$ *be projection onto the first column, and let* η_n *generate* $H^{2n-1,n}(\mathbf{A}^n \setminus 0, \mathbf{Z}) \cong \mathbf{Z}$. *Then* $w_n^* p_1^*(\eta_n) = \pm\sigma x^{n-1}$.

Proof. Part (1) follows from [21, Theorem 18], whose comparison uses unreduced cohomology on both sides. For $i \geq 2$, the coefficient group $H^{2i-1,i}(F, \mathbf{Z})$ vanishes, so the reduced and unreduced groups in this bidegree agree. The proof of that theorem shows that w_n^* sends the generator $\rho_n \in H^{2n-1,n}(GL_n, \mathbf{Z})$ to σx^{n-1}, and [21, Proposition 7] identifies ρ_n up to sign with $p_1^*(\eta_n)$. □

For $n \geq 2$, recall that $\pi_{n-1}^{\mathbf{A}^1}(\mathbf{A}^n \setminus 0) \cong \mathbf{K}_n^{MW}$, with canonical quotient map $\mathbf{K}_n^{MW} \to \mathbf{K}_n^M$.

Theorem 3.2 (Asok–Fasel–Williams)**.** *Let* F *be an infinite field and* $n \geq 2$. *The stabilization homomorphism* $\mathrm{st}_n\colon \pi_{n-1}^{\mathbf{A}^1}(GL_n)(F) \to K_n(F)$ *is surjective, and the composite*

$$\pi_{n-1}^{\mathbf{A}^1}(GL_n)(F) \xrightarrow{(p_1)_*} \pi_{n-1}^{\mathbf{A}^1}(\mathbf{A}^n \setminus 0)(F) = K_n^{MW}(F) \longrightarrow K_n^M(F)$$

equals $\pm h_n \circ \mathrm{st}_n$.

Proof. Over infinite fields, the affine singular complex computes $\pi_*^{\mathbf{A}^1}(GL_n)(F)$ [4, Proposition 2.3, Corollary 2.4, and Remark 2.5]. Because $\mathrm{fib}(BGL_a \to BGL_{a+1}) \simeq \mathbf{A}^{a+1} \setminus 0$ is $(a-1)$-connected for all $a \geq n$, the map st_n is surjective. The second statement is [4, Theorem 2.18]. □

Proposition 3.3. *For every infinite field* F *and every integer* $n \geq 2$, *we have* $c_{n,n} = \pm h_n$ *on* $K_n(F)$.

Proof. By Theorem 3.2, it suffices to check that $c_{n,n} \circ \mathrm{st}_n = \pm h_n \circ \mathrm{st}_n$ on $\pi_{n-1}^{\mathbf{A}^1}(GL_n)(F)$. Any pointed map $X \to \Omega^\infty \Sigma^{2n-1,n} H\mathbf{Z}$ representing a class in $\widetilde{H}^{2n-1,n}(X, \mathbf{Z})$ induces a sheaf morphism $\pi_{n-1}^{\mathbf{A}^1}(X) \to \pi_{n-1}^{\mathbf{A}^1}(\Omega^\infty \Sigma^{2n-1,n} H\mathbf{Z}) = \mathbf{K}_n^M$. Under $GL_n \simeq \Omega BGL_n \to \Omega BGL$, looping c_n gives $\Omega c_n \in H^{2n-1,n}(GL_n, \mathbf{Z})$, which induces $c_{n,n} \circ \mathrm{st}_n$ on $\pi_{n-1}^{\mathbf{A}^1}(GL_n)(F)$. For $X = \mathbf{A}^n \setminus 0$, contraction n times identifies both $H^{2n-1,n}(\mathbf{A}^n \setminus 0, \mathbf{Z})$ and $\mathrm{Hom}(\mathbf{K}_n^{MW}, \mathbf{K}_n^M)$ with $\mathbf{Z}$ [4, Lemma 3.14], so η_n induces ± 1 times the canonical quotient $\mathbf{K}_n^{MW} \to \mathbf{K}_n^M$, and $p_1^*(\eta_n) \in H^{2n-1,n}(GL_n, \mathbf{Z})$ induces $\pm h_n \circ \mathrm{st}_n$ by Theorem 3.2. By Theorem 3.1(1), it therefore suffices to show that $w_n^*(\Omega c_n) = \pm w_n^* p_1^*(\eta_n)$.

Cover $\mathbf{P}^1$ by its two standard affine charts with intersection $\mathbf{G}_m$, so $S^1 \wedge \mathbf{G}_m \simeq \mathbf{P}^1$. Gluing trivial bundles of rank n over $\mathbf{P}^1 \times J_{n-1}$ along the transition automorphism $w_n = \mathrm{id}_Q \oplus \lambda \cdot \mathrm{id}_L$ over $\mathbf{G}_m \times J_{n-1}$ identifies the map $\mathbf{P}^1 \wedge (J_{n-1})_+ \to BGL_n$ adjoint to w_n with the virtual clutching bundle $[Q \oplus (\mathcal{O}(\pm 1) \otimes L)] - [\mathcal{O}^n]$. Let $t = c_1(\mathcal{O}(1)) \in \widetilde{H}^{2,1}(\mathbf{P}^1, \mathbf{Z})$, so $t^2 = 0$. Since $Q \oplus L \cong \mathcal{O}^n$ gives $c(Q) = (1+x)^{-1}$, the total Chern class of this virtual bundle is

$$\frac{1 + x \pm t}{1 + x} = 1 \pm \frac{t}{1+x} = 1 \pm \sum_{i \geq 1} (-1)^{i-1} t x^{i-1},$$

whose degree-n term is $\pm(-1)^{n-1} t x^{n-1}$. Passing back across the $(S^1 \wedge -, \Omega)$-adjunction sends t to $\pm\sigma$ and gives $w_n^*(\Omega c_n) = \pm\sigma x^{n-1}$, which agrees up to sign with $w_n^* p_1^*(\eta_n)$ by Theorem 3.1(2). □

3.2. **Stabilization on** $f_m\mathrm{KGL}$**.** Let F be a perfect field of exponential characteristic e. The connected infinite loop space $BGL \simeq \Omega^\infty f_1 \mathrm{KGL}$ is $\mathbf{A}^1$-nilpotent [3, Example 3.4.1]. Localization of the spectrum localizes its homotopy sheaves, so $\Omega^\infty f_1 \mathrm{KGL}[1/e]$ is a connected infinite loop space whose $\mathbf{A}^1$-homotopy sheaves are those of BGL tensored with $\mathbf{Z}[1/e]$. The localization criterion [3, Theorem 4.3.9 and Corollary 4.3.10] therefore identifies it with $BGL[1/e]$. Thus c_n extends uniquely to a pointed map $c_n\colon BGL[1/e] \to \Omega^\infty \Sigma^{2n,n} H\mathbf{Z}[1/e]$.

Theorem 3.4 (Asok–Bachmann–Elmanto–Hopkins)**.** *Let* F *be a perfect field of exponential characteristic* e, *let* $m \geq 2$, *and let* $E \in \Sigma^{2m,m}\mathrm{SH}(F)^{\mathrm{veff}}[1/e]$. *Then* $\mathrm{fib}(\Sigma^\infty \Omega^\infty E \to E)$ *lies in* $\Sigma^{4m,2m}\mathrm{SH}(F)^{\mathrm{veff}}[1/e]$.

Proof. This is the range $m \geq 2$ of [1, Lemma 7.19], obtained from the infinite-loop $\mathbf{P}^1$-Freudenthal theorem [2, Theorem 6.3.3]. In the notation of that theorem, $p = 2m$ and $q = m$, so both $q \geq 2$ and $p - q \geq 2$ hold. □

Proposition 3.5. *Let F be a perfect field of exponential characteristic e, let $n \geq 3$, and put $m = \lfloor n/2 \rfloor + 1$. The restriction of c_n along $\Omega^\infty f_m \mathrm{KGL}[1/e] \to BGL[1/e]$ equals $\Omega^\infty C_n$ for a unique morphism*

$$C_n \colon f_m \mathrm{KGL}[1/e] = \Sigma^{2m,m}\mathrm{kgl}[1/e] \longrightarrow \Sigma^{2n,n} H\mathbf{Z}[1/e]$$

in $\mathrm{SH}(F)[1/e]$, and the formation of C_n commutes with base change along extensions of perfect fields.

Proof. Set $B = \mathrm{fib}(\Sigma^\infty\Omega^\infty\Sigma^{2m,m}\mathrm{kgl}[1/e] \to \Sigma^{2m,m}\mathrm{kgl}[1/e])$. Since kgl is very effective, Theorem 3.4 shows that B and $\Sigma^{1,0}B$ are $2m$-effective. Because $\Sigma^{2n,n}H\mathbf{Z}[1/e]$ is an n-slice and $n < 2m$, both $[B, \Sigma^{2n,n}H\mathbf{Z}[1/e]]$ and $[\Sigma^{1,0}B, \Sigma^{2n,n}H\mathbf{Z}[1/e]]$ vanish, so the $(\Sigma^\infty, \Omega^\infty)$-adjunction gives a bijection

$$[\Sigma^{2m,m}\mathrm{kgl}[1/e], \Sigma^{2n,n}H\mathbf{Z}[1/e]] \xrightarrow{\sim} [\Omega^\infty\Sigma^{2m,m}\mathrm{kgl}[1/e], \Omega^\infty\Sigma^{2n,n}H\mathbf{Z}[1/e]]_*.$$

This bijection produces C_n and its uniqueness. Since f_m, Ω^∞, and c_n commute with base change along extensions of perfect fields (Lemma 2.2), uniqueness gives the final claim. □

3.3. **Bott-shift recurrences and proof of Theorem B.** For $s \geq 1$, applying the functor $\Omega^s_{\mathbf{P}^1}\Omega^\infty\Sigma^{2,1} \simeq \Omega^\infty\Sigma^{2-2s,1-s}$ to $\beta^s \colon \Sigma^{2s,s}\mathrm{kgl} \to \mathrm{kgl}$ gives a pointed map

$$BGL = \Omega^\infty f_1 \mathrm{KGL} \simeq \Omega^s_{\mathbf{P}^1}\Omega^\infty f_{s+1}\mathrm{KGL} \xrightarrow{\Omega^s_{\mathbf{P}^1}\Omega^\infty(\Sigma^{2,1}\beta^s)} \Omega^s_{\mathbf{P}^1} BGL,$$

whose adjoint $(\mathbf{P}^1)^{\wedge s} \wedge BGL \to BGL$ classifies external multiplication $V \mapsto V \cdot z_s$ by the s-fold reduced Bott class

$$z_s = \prod_{a=1}^{s} ([\mathcal{O}(1)_a] - 1) \in \widetilde{K}_0((\mathbf{P}^1)^{\wedge s}).$$

Set $t_a = c_1(\mathcal{O}(1)_a) \in \widetilde{H}^{2,1}(\mathbf{P}^1, \mathbf{Z})$, so $t_a^2 = 0$, put $U_s = t_1 \cdots t_s \in \widetilde{H}^{2s,s}((\mathbf{P}^1)^{\wedge s}, \mathbf{Z})$, and write $c_v = \sum_{i \geq 0} c_i v^i$ for the total Chern class in a formal variable v. For a line bundle M on a smooth scheme X with $c_1(M) = x$, the virtual bundle

$$M \cdot z_s = \sum_{S \subseteq \{1,\dots,s\}} (-1)^{s-|S|} \Big[M \otimes \bigotimes_{a \in S} \mathcal{O}(1)_a \Big]$$

has total Chern class

$$D_s(x) = \prod_{S \subseteq \{1,\dots,s\}} \Big(1 + v\big(x + \textstyle\sum_{a \in S} t_a\big)\Big)^{(-1)^{s-|S|}}.$$

Lemma 3.6. *With the notation above, let $s \geq 1$.*

(1) $D_s(x) = 1 + (-1)^{s-1}(s-1)!\, v^s U_s (1+vx)^{-s}$.
(2) For $n \geq 1$, we have $c_n(z_n) = (-1)^{n-1}(n-1)!\, U_n$.
(3) For $n > s$ and every virtual bundle V on X,

$$c_n(V \cdot z_s) = (-1)^{n-1} \frac{(n-1)!}{(n-s)!}\, U_s\, p_{n-s}(V),$$

where p_h denotes the hth power-sum polynomial in the Chern roots, expressed in the Chern classes of V via Newton's identities.

Proof.

(1) For $s = 1$, we have $D_1(x) = (1 + v(x + t_1))/(1 + vx) = 1 + vt_1(1+vx)^{-1}$. Separating the subsets $S \subseteq \{1, \dots, s+1\}$ containing $s+1$ from the rest gives $D_{s+1}(x) = D_s(x + t_{s+1})/D_s(x)$. If $D_s(x) = 1 + cU_s(1+vx)^{-s}$, then $t_{s+1}^2 = 0$ and $U_s^2 = 0$ give

$$D_s(x + t_{s+1}) = 1 + cU_s\big((1+vx)^{-s} - svt_{s+1}(1+vx)^{-s-1}\big)$$

and thus $D_{s+1}(x) = 1 - scv\,U_{s+1}(1+vx)^{-s-1}$. Induction on s gives (1).

(2) Setting $s = n$ and $x = 0$ in (1) gives (2).

(3) By the splitting principle, write $V = \sum_i \epsilon_i[M_i]$ with $\epsilon_i = \pm 1$ and $c_1(M_i) = x_i$. Since $U_s^2 = 0$, part (1) gives

$$c_v(V \cdot z_s) = \prod_i D_s(x_i)^{\epsilon_i} = 1 + (-1)^{s-1}(s-1)!\, v^s U_s \sum_i \epsilon_i (1+vx_i)^{-s}.$$

The coefficient of v^{n-s} in $(1+vx_i)^{-s}$ is $(-1)^{n-s}\binom{n-1}{n-s}x_i^{n-s}$, and $\sum_i \epsilon_i x_i^{n-s} = p_{n-s}(V)$. Since $(s-1)!\binom{n-1}{n-s} = (n-1)!/(n-s)!$, comparing the coefficients of v^n gives (3). □

Because $\bigoplus_{i\geq 0} H^{2i,i}(BGL, \mathbf{Z}) = \mathbf{Z}[c_1, c_2, \ldots]$ [7, §6.1], the identities of Lemma 3.6 hold universally on BGL and extend by localization to $BGL[1/e]$. For $n \geq 3$ with m, r as in (1.1) and a prime field k of exponential characteristic e, let C_n be the morphism of Proposition 3.5 over k, and define

$$c_n^{\mathrm{st}} = \Sigma^{-2m,-m} C_n \colon \mathrm{kgl}[1/e] \longrightarrow \Sigma^{2r,r} H\mathbf{Z}[1/e].$$

Proposition 3.7. *Over a prime field k of exponential characteristic e, the operations c_n^{st} satisfy:*

(1) $\Omega^\infty(\Sigma^{2,1}c_3^{\mathrm{st}}) = \pm p_2 = \pm(c_1^2 - 2c_2)$ *in* $\widetilde{H}^{4,2}(BGL[1/e], \mathbf{Z}[1/e])$.

(2) $c_{2r+2}^{\mathrm{st}} = \pm(2r+1)\, c_{2r+1}^{\mathrm{st}}$ *in* $[\mathrm{kgl}[1/e], \Sigma^{2r,r}H\mathbf{Z}[1/e]]$ *for every* $r \geq 1$.

(3) $\beta^*(c_{2r+1}^{\mathrm{st}}) = \pm 2r(2r-1)\, c_{2r-1}^{\mathrm{st}}$ *in* $[\mathrm{kgl}[1/e], \Sigma^{2r-2,r-1}H\mathbf{Z}[1/e]]$ *for every* $r \geq 2$.

Consequently, $(\beta^*)^{r-1}(c_n^{\mathrm{st}}) = \pm\frac{(n-1)!}{2}\, c_3^{\mathrm{st}}$ *for every* $n \geq 3$ *with* $r = \lceil n/2 \rceil - 1$.

Proof. For integers $n > s \geq 1$, let $\Phi_{n,s}$ denote the composite

$$BGL[1/e] \simeq \Omega^s_{\mathbf{P}^1}\Omega^\infty f_{s+1}\mathrm{KGL}[1/e] \xrightarrow{\Omega^s_{\mathbf{P}^1}\Omega^\infty(\Sigma^{2,1}\beta^s)} \Omega^s_{\mathbf{P}^1}BGL[1/e] \xrightarrow{\Omega^s_{\mathbf{P}^1}c_n} \Omega^\infty\Sigma^{2(n-s),n-s}H\mathbf{Z}[1/e].$$

Adjointly on $(\mathbf{P}^1)^{\wedge s} \wedge BGL$, pulling back c_n gives $c_n(V \cdot z_s) = (-1)^{n-1}\frac{(n-1)!}{(n-s)!}\, U_s\, p_{n-s}(V)$ by Lemma 3.6(3), so passing across the $((\mathbf{P}^1)^{\wedge s} \wedge -, \Omega^s_{\mathbf{P}^1})$-adjunction gives

$$\Phi_{n,s} = \pm\frac{(n-1)!}{(n-s)!}\, p_{n-s} \qquad \text{in } \widetilde{H}^{2(n-s),n-s}(BGL[1/e], \mathbf{Z}[1/e]).$$

For $(n,s) = (3,1)$, we have $\Sigma^{2,1}c_3^{\mathrm{st}} = \Sigma^{-2,-1}C_3$ and $\Omega^\infty(\Sigma^{2,1}c_3^{\mathrm{st}}) = \Omega_{\mathbf{P}^1}(\Omega^\infty C_3) = \Phi_{3,1} = \pm p_2$, proving (1).

Next, fix $a \geq 2$ and let $\iota_a = \Omega^\infty(\Sigma^{2,1}\beta^{a-1}) \colon \Omega^\infty f_a\mathrm{KGL}[1/e] \to BGL[1/e]$. By Theorem 3.4, $\mathrm{fib}(\Sigma^\infty\Omega^\infty f_a\mathrm{KGL}[1/e] \to f_a\mathrm{KGL}[1/e])$ is $2a$-effective, so the $(\Sigma^\infty, \Omega^\infty)$-adjunction gives a bijection

$$[f_a\mathrm{KGL}[1/e], \Sigma^{2i,i}H\mathbf{Z}[1/e]] \xrightarrow{\sim} \widetilde{H}^{2i,i}(\Omega^\infty f_a\mathrm{KGL}[1/e], \mathbf{Z}[1/e]) \qquad (1 \leq i < 2a).$$

When $1 \leq i < a$, the source $f_a\mathrm{KGL}[1/e]$ is a-effective while $\Sigma^{2i,i}H\mathbf{Z}[1/e]$ is an i-slice, so the left-hand group vanishes and $\iota_a^*(c_i) = 0$. By Newton's identities, $p_{2a-1} \equiv (2a-1)c_{2a-1}$ modulo products $c_{i_1}\cdots c_{i_t}$ with $t \geq 2$, $i_1 \leq \cdots \leq i_t$, and $i_1 + \cdots + i_t = 2a-1$. Every such product satisfies $2i_1 \leq 2a-1$, hence $i_1 < a$, and therefore vanishes under the ring homomorphism ι_a^*. Thus $\iota_a^*(p_{2a-1}) = (2a-1)\iota_a^*(c_{2a-1}) = (2a-1)\Omega^\infty C_{2a-1}$.

For (2), let $r \geq 1$, so $\Sigma^{2r+2,r+1}c_{2r+2}^{\mathrm{st}} = \Sigma^{-2,-1}C_{2r+2}$. Factoring $\iota_{r+2} = \Omega^\infty(\Sigma^{2,1}\beta) \circ \Omega^\infty(\Sigma^{4,2}\beta^r)$ and applying $\Omega^\infty\Sigma^{-2,-1} \simeq \Omega_{\mathbf{P}^1}\Omega^\infty$ gives

$$\Omega^\infty(\Sigma^{2r+2,r+1}c_{2r+2}^{\mathrm{st}}) = \Phi_{2r+2,1} \circ \iota_{r+1} = \pm\iota_{r+1}^*(p_{2r+1}) = \pm(2r+1)\Omega^\infty C_{2r+1}.$$

Bijectivity of Ω^∞ for $a = r+1$ and $i = 2r+1 < 2a$ gives $\Sigma^{2r+2,r+1}c_{2r+2}^{\mathrm{st}} = \pm(2r+1)C_{2r+1}$, proving (2).

For (3), let $r \geq 2$, so $\Sigma^{2r,r}\beta^*(c^{\mathrm{st}}_{2r+1}) = \Sigma^{-4,-2}(C_{2r+1} \circ \Sigma^{2r+2,r+1}\beta)$. On $\Omega^\infty f_{r+2}\mathrm{KGL}[1/e]$, the morphism $C_{2r+1} \circ \Sigma^{2r+2,r+1}\beta$ induces $c_{2r+1} \circ \iota_{r+2} = c_{2r+1} \circ \Omega^\infty(\Sigma^{2,1}\beta^2) \circ \Omega^\infty(\Sigma^{6,3}\beta^{r-1})$. Applying $\Omega^\infty\Sigma^{-4,-2} \simeq \Omega^2_{\mathbf{P}^1}\Omega^\infty$ gives

$$\Omega^\infty(\Sigma^{2r,r}\beta^*(c^{\mathrm{st}}_{2r+1})) = \Phi_{2r+1,2} \circ \iota_r = \pm 2r\, \iota_r^*(p_{2r-1}) = \pm 2r(2r-1)\Omega^\infty C_{2r-1}.$$

Bijectivity of Ω^∞ for $a = r$ and $i = 2r-1 < 2r$ gives $\Sigma^{2r,r}\beta^*(c^{\mathrm{st}}_{2r+1}) = \pm 2r(2r-1)C_{2r-1}$, proving (3). Induction on r using (2) and (3) gives the final formula. □

Proof of Theorem B.

(1) Applying $\Sigma^{2m,m}$ identifies the Bott value of c^{st}_n with the composite

$$S^{2n,n} \longrightarrow \Sigma^{2n,n}\mathrm{kgl} \xrightarrow{\Sigma^{2m,m}\beta^r} \Sigma^{2m,m}\mathrm{kgl} \xrightarrow{C_n} \Sigma^{2n,n}H\mathbf{Z}[1/e].$$

Unstably, $S^{2n,n} \to \Sigma^{2m,m}\mathrm{kgl} = f_m\mathrm{KGL}$ represents $z_n \in \widetilde{K_0}((\mathbf{P}^1)^{\wedge n})$, on which $\Omega^\infty C_n = c_n|_{\Omega^\infty f_m\mathrm{KGL}[1/e]}$ takes the value $c_n(z_n) = \pm(n-1)!\,U_n$ by Lemma 3.6(2). Hence c^{st}_n has Bott value $\pm(n-1)!$, and the second formula in (1) is Proposition 3.7.

(2) Let F/k be an infinite perfect field and let $y \in \pi_{n,0}(f_m\mathrm{KGL})(F)$ have image $\bar{y} \in K_n(F)$. Under $f_m\mathrm{KGL} \simeq \Sigma^{2m,m}\mathrm{kgl}$, base-change compatibility of C_n (Proposition 3.5) and Proposition 3.3 give

$$c^{\mathrm{st}}_n(y) = C_n(y) = c_{n,n}(\bar{y}) = \pm h_n(\bar{y}) \qquad \text{in } K^M_n(F)[1/e],$$

since $\Omega^\infty C_n = c_n|_{\Omega^\infty f_m\mathrm{KGL}[1/e]}$.

(3) Part (3) is Proposition 2.4. □

4. The Bott tower

Fix $n \geq 3$ with m, r, j as in (1.1), together with k, ℓ, and $q = \ell^\nu$ satisfying the hypotheses of Theorem C. For $s \geq 1$, let k_s be the minimal extension of k over which $\mu_q^{\otimes s}$ is constant: if $k = \mathbf{Q}$, then $k_s = \mathbf{Q}(\mu_q)^{\ker(\chi_q^s)}$ for the mod-q cyclotomic character χ_q; if $k = \mathbf{F}_p$, then $k_s = \mathbf{F}_{p^d}$ where d is the multiplicative order of p^s in $(\mathbf{Z}/q)^\times$. Choose a generator τ_s of $H^{0,s}(k_s, \mathbf{Z}/q) = H^0_{\mathrm{et}}(k_s, \mu_q^{\otimes s}) \cong \mathbf{Z}/q$.

Lemma 4.1. *With k, ℓ, q as in Theorem C, for every $s \geq 1$ there exists $\mathfrak{b}_s \in \pi_{0,-s}(\mathrm{kgl}/q)(k_s)$ such that $u(\mathfrak{b}_s) = \tau_s$.*

Proof. If $k = \mathbf{F}_p$, then $\mathrm{cd}_\ell(k_s) = 1$, and Lemma 2.6 gives the statement directly. Suppose $k = \mathbf{Q}$, so ℓ is odd. Kahn's construction [11, Corollary 9.5(0) and (v)] supplies a Bott class in $K_{2s}(k_s; \mathbf{Z}/q)$ whose restriction to $\bar{\mathbf{Q}}$ has order q, and Lemma 2.5 lifts it uniquely to $\pi_{0,-s}(\mathrm{kgl}/q)(k_s)$. Since $\mathrm{cd}_\ell(\bar{\mathbf{Q}}) = 0$, Lemma 2.6 shows that the edge of this lift generates $H^{0,s}(\bar{\mathbf{Q}}, \mathbf{Z}/q)$. Because k_s splits $\mu_q^{\otimes s}$, restriction $H^{0,s}(k_s, \mathbf{Z}/q) \to H^{0,s}(\bar{\mathbf{Q}}, \mathbf{Z}/q)$ is an isomorphism, so the edge over k_s is a unit multiple of τ_s in $\mathbf{Z}/q$. Multiplying this lift by the inverse unit in $(\mathbf{Z}/q)^\times$ gives $\mathfrak{b}_s$. □

We call $\mathfrak{b}_s$ a *Bott element* and write $\mathfrak{b}_s$ also for its restriction to any field over k_s.

Theorem 4.2 (Kahn)**.** *Let k, ℓ, q be as in Theorem C, let F be a field over k, let $s \geq 1$, and let Fk_s be the compositum inside a separable closure of F. For every $a \geq 1$, corestriction induces an isomorphism*

$$\mathrm{cor}\colon H^a_{\mathrm{et}}(Fk_s, \mu_q^{\otimes(a+s)})_{\mathrm{Gal}(Fk_s/F)} \xrightarrow{\sim} H^a_{\mathrm{et}}(F, \mu_q^{\otimes(a+s)}).$$

Proof. This is [10, Definition 1 and Theorem 1(2)] applied to $\mu_q^{\otimes(a+s)}$ in degree a, since Fk_s/F is the minimal extension splitting $\mu_q^{\otimes s}$. Kahn's liftability condition (R) holds because the cyclotomic character lifts to modulus $\ell^{2\nu}$, and symbol surjectivity over Fk_s holds by the norm residue theorem [20]. When $\ell = 2$, we have $k = \mathbf{F}_p$, and it remains to verify that $-1 \notin \mathrm{im}(\chi_q^s) \subseteq \langle p^s \rangle \subseteq (\mathbf{Z}/2^\nu)^\times$. If $p \equiv 1 \pmod 4$ or s is even, then $p^s \equiv 1 \pmod 4$, whereas $-1 \not\equiv 1 \pmod 4$ since $\nu \geq 3$. If $p \equiv 3 \pmod 4$ and s is odd, then $p^{si} \equiv 1 \not\equiv -1$

(mod 4) for even i, while $v_2(p^{si}+1) = v_2(p+1) < \nu$ for odd i, so $p^{si} \not\equiv -1 \pmod{2^\nu}$ for all $i \geq 0$. □

Remark 4.3. The bound $\nu > v_2(p+1)$ in Theorem 4.2 cannot be dropped. For $F = \mathbf{F}_7((z))$, $q = 8$, and $s = 1$, we have $k_1 = \mathbf{F}_{49}$ and $Fk_1 = \mathbf{F}_{49}((z))$. The residue map $H^1_{\mathrm{et}}(F, \mu_8^{\otimes 2}) \to H^0_{\mathrm{et}}(\mathbf{F}_7, \mu_8)$ sends $\{z\} \cup (-1)$ to $-1 \neq 0$, whereas the residue of any corestricted class lies in $\mathrm{im}(N_{\mathbf{F}_{49}/\mathbf{F}_7} \colon \mu_8 \to \mu_8) = \mu_8^8 = \{1\}$.

Proposition 4.4. *Let $n \geq 3$ with m, r, j as in (1.1), and let k, ℓ, q be as in Theorem C. For every perfect field F over k, every class $x \in \pi_{-j,-m}(\mathrm{kgl}/q)(F)$ can be written as*

$$x = \sum_{s=1}^{r} \beta^{r-s} \mathrm{Tr}_{Fk_s/F}(\mathfrak{b}_s \cdot \widehat{A}_s) + \beta^r x_0, \tag{4.1}$$

where $\widehat{A}_s \in \pi_{-(n-2s),-(n-2s)}\mathbf{1}(Fk_s)$ is a sphere lift of some $A_s \in K^M_{n-2s}(Fk_s)$ and $x_0 \in \pi_{-n,-n}(\mathrm{kgl}/q)(F)$.

Proof. Set $x_r = x$. Given $x_s \in \pi_{-(n-2s),-(n-s)}(\mathrm{kgl}/q)(F)$ with $1 \leq s \leq r$, we have $1 \leq j \leq n-2s < n-s$, so $u(x_s)$ lies in $H^{n-2s,n-s}(F, \mathbf{Z}/q) \cong H^{n-2s}_{\mathrm{et}}(F, \mu_q^{\otimes(n-s)})$. Over Fk_s, the sheaf $\mu_q^{\otimes s}$ is constant with generator τ_s, so cup product with τ_s and the norm residue theorem give isomorphisms

$$K^M_{n-2s}(Fk_s)/q \xrightarrow{\sim} H^{n-2s}_{\mathrm{et}}(Fk_s, \mu_q^{\otimes(n-2s)}) \xrightarrow[\sim]{\tau_s \cup -} H^{n-2s}_{\mathrm{et}}(Fk_s, \mu_q^{\otimes(n-s)}).$$

By Theorem 4.2, there exists $A_s \in K^M_{n-2s}(Fk_s)$ with mod-q reduction $\bar{A}_s$ such that $u(x_s) = \mathrm{cor}(\tau_s \cup \bar{A}_s)$. Choose a sphere lift $\widehat{A}_s$ of A_s. By Lemma 4.1 and compatibility of u with $\pi_{*,*}\mathbf{1}$-products and transfers, $\mathrm{Tr}_{Fk_s/F}(\mathfrak{b}_s \cdot \widehat{A}_s)$ also has edge $\mathrm{cor}(\tau_s \cup \bar{A}_s)$. Exactness of (2.1) therefore gives

$$x_s - \mathrm{Tr}_{Fk_s/F}(\mathfrak{b}_s \cdot \widehat{A}_s) = \beta x_{s-1} \qquad \text{for some } x_{s-1} \in \pi_{-(n-2(s-1)),-(n-(s-1))}(\mathrm{kgl}/q)(F).$$

Repeating this step r times produces (4.1). □

Proof of Theorem C. For $0 \leq s \leq r$, set $\alpha_s = (\beta^*)^{r-s}(c_n^{\mathrm{st}}) \colon \mathrm{kgl}[1/e] \to \Sigma^{2s,s} H\mathbf{Z}[1/e]$ in $\mathrm{SH}(k)[1/e]$, so $\alpha_r = c_n^{\mathrm{st}}$ and every α_s has Bott value $\pm(n-1)!$ by Theorem B(1). By Lemma 2.1, we have $\alpha_0 = \pm(n-1)!\, u$. For every $x_0 \in \pi_{-n,-n}(\mathrm{kgl}/q)(F)$, we have $c_n^{\mathrm{st}}(\beta^r x_0) = \alpha_0(x_0) = \pm(n-1)!\, u(x_0)$, so $\mathrm{im}(c_n^{\mathrm{st}})$ contains $(n-1)!\, K^M_n(F)/q$ by (2.4).

Conversely, let $x \in \pi_{-j,-m}(\mathrm{kgl}/q)(F)$ and decompose x as in (4.1). Because α_s is defined over k, it commutes with $\mathrm{Tr}_{Fk_s/F}$ and $\pi_{*,*}\mathbf{1}$-multiplication, so

$$c_n^{\mathrm{st}}(x) = \sum_{s=1}^{r} \mathrm{Tr}_{Fk_s/F}(\alpha_s(\mathfrak{b}_s) \cdot \bar{A}_s) \pm (n-1)!\, u(x_0) \qquad \text{in } K^M_n(F)/q,$$

where $\alpha_s(\mathfrak{b}_s)$ is restricted from $\alpha_s(\mathfrak{b}_s) \in K^M_{2s}(k_s)/q \cong H^{2s}_{\mathrm{et}}(k_s, \mu_q^{\otimes 2s})$. For $1 \leq s \leq r$, this group vanishes if $k = \mathbf{F}_p$ (since $\mathrm{cd}_\ell(k_s) = 1 < 2s$) and if $k = \mathbf{Q}$ and $s \geq 2$ (since ℓ is odd and $\mathrm{cd}_\ell(k_s) = 2 < 2s$). When $k = \mathbf{Q}$ and $s = 1$, Theorem B(1) gives $\alpha_1 = \pm\frac{(n-1)!}{2} c_3^{\mathrm{st}}$; since ℓ is odd, 2 is invertible in $\mathbf{Z}_{(\ell)}$, so $\alpha_1(\mathfrak{b}_1) \in (n-1)!\, K^M_2(k_1)/q$. Thus $c_n^{\mathrm{st}}(x) \in (n-1)!\, K^M_n(F)/q$. □

5. Two-local operations in characteristic zero

Write $H = H\mathbf{Z}/2$, $\mathrm{red}_2 \colon H\mathbf{Z}_{(2)} \to H$ for mod-2 reduction, and $\partial_2 \colon H \to \Sigma^{1,0} H\mathbf{Z}_{(2)}$ for the 2-local integral Bockstein, so $\mathrm{red}_2 \circ \partial_2 = Sq^1$. Let $\mathcal{A}$ be the mod-2 motivic Steenrod algebra over $\mathbf{Q}$, which is free as a left $H^{*,*}(\mathbf{Q}, \mathbf{Z}/2)$-module on the admissible monomials Sq^I [9, Theorem 1.1]. Over $\mathbf{Q}$, for every $s \geq 1$, the 2-localized Bott cofiber sequence (2.1) induces an exact sequence

$$\begin{aligned}[H\mathbf{Z}_{(2)}, \Sigma^{2s,s} H\mathbf{Z}_{(2)}] &\xrightarrow{u^*} [\mathrm{kgl}_{(2)}, \Sigma^{2s,s} H\mathbf{Z}_{(2)}] \\ &\xrightarrow{\beta^*} [\mathrm{kgl}_{(2)}, \Sigma^{2s-2,s-1} H\mathbf{Z}_{(2)}] \xrightarrow{\delta_0^*} [H\mathbf{Z}_{(2)}, \Sigma^{2s+1,s} H\mathbf{Z}_{(2)}].\end{aligned} \tag{5.1}$$

5.1. **Integral cohomology operations.**

Lemma 5.1. *Over* $\mathbf{Q}$*, for every* $b > 0$ *and* $a \in \mathbf{Z}$*, the kernel* K *of restriction along the unit,*

$$[H\mathbf{Z}_{(2)}, \Sigma^{a,b} H\mathbf{Z}_{(2)}] \longrightarrow H^{a,b}(\mathbf{Q}, \mathbf{Z}_{(2)}),$$

is killed by 2*. In particular,* $[H\mathbf{Z}_{(2)}, \Sigma^{2s,s} H\mathbf{Z}_{(2)}]$ *and* $[H\mathbf{Z}_{(2)}, \Sigma^{2s+1,s} H\mathbf{Z}_{(2)}]$ *over* $\mathbf{Q}$ *are killed by* 2 *for all* $s > 0$*.*

Proof. The module action of $H^{*,*}(\mathbf{Q}, \mathbf{Z}_{(2)})$ on $H\mathbf{Z}_{(2)}$ splits restriction along $\mathbf{1} \to H\mathbf{Z}_{(2)}$ compatibly with red_2 and ∂_2, so K is a direct summand.

(1) Since $s_0\mathbf{1}_{(2)} \simeq H\mathbf{Z}_{(2)}$, the cofiber sequence $f_1\mathbf{1}_{(2)} \to \mathbf{1}_{(2)} \to H\mathbf{Z}_{(2)}$ identifies K with the image of $[\Sigma^{1,0} f_1\mathbf{1}_{(2)}, \Sigma^{a,b} H\mathbf{Z}_{(2)}]$. Because $\Sigma^{a,b} H\mathbf{Z}_{(2)}$ is a b-slice, maps into it from shifts of $f_{b+1}\mathbf{1}_{(2)}$ vanish, so $f_1\mathbf{1}_{(2)}$ may be replaced by $f_1\mathbf{1}_{(2)}/f_{b+1}\mathbf{1}_{(2)}$. By [17, Theorem 2.12 and Appendix B], each slice $s_w\mathbf{1}_{(2)}$ $(1 \le w \le b)$ is a finite wedge of suspensions of motivic Eilenberg–Mac Lane spectra with coefficients in $\mathrm{Ext}^{i,2w}_{MU_*MU}(MU_*, MU_*)_{(2)}$. For fixed $w > 0$, these Ext-groups are nonzero for only finitely many i, finitely generated in each bidegree, and rationally zero because the logarithm makes every formal group law over a $\mathbf{Q}$-algebra strictly isomorphic to the additive law; thus each coefficient group is a finite 2-group. Since $2 \cdot \mathrm{id}_H = 0$, every finite 2-primary motivic Eilenberg–Mac Lane spectrum has identity map killed by a power of 2, so K is killed by 2^k for some $k \ge 1$.

(2) Since $2 \cdot \mathrm{id}_H = 0$, applying $[-, \Sigma^{*,*}H]$ to $H\mathbf{Z}_{(2)} \xrightarrow{2} H\mathbf{Z}_{(2)} \xrightarrow{\mathrm{red}_2} H \xrightarrow{\partial_2} \Sigma^{1,0} H\mathbf{Z}_{(2)}$ gives a short exact sequence

$$0 \longrightarrow [\Sigma^{1,0} H\mathbf{Z}_{(2)}, \Sigma^{*,*}H] \xrightarrow{\partial_2^*} [H, \Sigma^{*,*}H] \xrightarrow{\mathrm{red}_2^*} [H\mathbf{Z}_{(2)}, \Sigma^{*,*}H] \longrightarrow 0.$$

Surjectivity of red_2^* on $[\Sigma^{1,0} H\mathbf{Z}_{(2)}, \Sigma^{*,*}H]$ identifies $\mathrm{im}(\partial_2^*)$ with the image of precomposition by $\mathrm{red}_2 \circ \partial_2 = Sq^1$, which is the left principal ideal $\mathcal{A}Sq^1$, so

$$[H\mathbf{Z}_{(2)}, \Sigma^{*,*}H] \cong \mathcal{A}/\mathcal{A}Sq^1. \tag{5.2}$$

For an admissible sequence $I = (i_1, \dots, i_t)$, right multiplication by Sq^1 sends Sq^I to the admissible monomial $Sq^{(i_1,\dots,i_t,1)}$ when $i_t \ge 2$ and to 0 when $i_t = 1$. Hence the reduced summand of $\mathcal{A}/\mathcal{A}Sq^1$ is free as a left $H^{*,*}(\mathbf{Q}, \mathbf{Z}/2)$-module on nonempty admissible monomials $Sq^{i_1} \cdots Sq^{i_t}$ with $i_t \ge 2$ (which forces $i_1 \ge 2$). Because $Sq^1 Sq^{2c} = Sq^{2c+1}$, $Sq^1 Sq^{2c+1} = 0$, and $2i_2 \le 2c+1$ implies $2i_2 \le 2c$ for $c \ge 1$, these basis monomials partition into admissible pairs

$$(M_{\mathrm{even}}, M_{\mathrm{odd}}) = \left(Sq^{2c} Sq^{I'},\ Sq^{2c+1} Sq^{I'}\right)$$

satisfying $Sq^1 M_{\mathrm{even}} = M_{\mathrm{odd}}$ and $Sq^1 M_{\mathrm{odd}} = 0$ under left multiplication by Sq^1. Define h linearly over $H^{*,*}(\mathbf{Q}, \mathbf{Z}/2)$ on the reduced summand of $\mathcal{A}/\mathcal{A}Sq^1$ by $h(M_{\mathrm{even}}) = 0$ and $h(M_{\mathrm{odd}}) = M_{\mathrm{even}}$. Because the Bockstein Sq^1 satisfies $Sq^1(c \cdot M) = Sq^1(c) \cdot M + c \cdot Sq^1(M)$ for $c \in H^{*,*}(\mathbf{Q}, \mathbf{Z}/2)$, both $c \cdot M_{\mathrm{even}}$ and $c \cdot M_{\mathrm{odd}}$ satisfy $(Sq^1 h + h Sq^1)(z) = z$.

(3) Let $x \in K$. Postcomposing $\mathrm{red}_2 \circ x$ with $Sq^1 = \mathrm{red}_2 \circ \partial_2$ gives $\mathrm{red}_2 \circ (\partial_2 \circ \mathrm{red}_2) \circ x = 0$, so step (2) produces a reduced class $y \in [H\mathbf{Z}_{(2)}, \Sigma^{a-1,b} H]$ with $\mathrm{red}_2 \circ x = Sq^1 \circ y = \mathrm{red}_2 \circ (\partial_2 \circ y)$. Projecting $x - \partial_2 \circ y$ onto the direct summand K gives $x - \partial_2 \circ y = 2z$ for some $z \in K$. Since $2\partial_2 = 0$, we obtain $2x = 4z$, so $2K = 4K = \cdots = 2^k K = 0$ by step (1).

For $s > 0$, both $H^{2s,s}(\mathbf{Q}, \mathbf{Z}_{(2)})$ and $H^{2s+1,s}(\mathbf{Q}, \mathbf{Z}_{(2)})$ vanish because the degree exceeds the weight. The final statement follows. □

5.2. **Weight-one operations and the operations θ_s.**

Lemma 5.2. *Over* $\mathbf{Q}$*:*

(1) $[H\mathbf{Z}_{(2)}, \Sigma^{2,1}H\mathbf{Z}_{(2)}] = 0$, $[H\mathbf{Z}_{(2)}, \Sigma^{2,1}H] = \mathbf{F}_2 \cdot (Sq^2 \operatorname{red}_2)$, *and* $[H\mathbf{Z}_{(2)}, \Sigma^{3,1}H] = \mathbf{F}_2 \cdot (Sq^3 \operatorname{red}_2)$.

(2) $[\mathrm{kgl}_{(2)}, \Sigma^{3,1}H] = 0$*; in particular,* $Q_1 \operatorname{red}_2 u = 0$*, where* $Q_1 = Sq^1Sq^2 + Sq^2Sq^1$.

(3) On the E_1*-page of the* 2*-local slice spectral sequence of* KGL*, the first differential satisfies* $\operatorname{red}_2 \circ d_1 = Sq^3 \circ \operatorname{red}_2$.

(4) $\operatorname{red}_2 c_3^{\mathrm{st}} = Sq^2 \operatorname{red}_2 u$ *in* $[\mathrm{kgl}_{(2)}, \Sigma^{2,1}H]$.

Proof.

(1) By (5.2), $[H\mathbf{Z}_{(2)}, \Sigma^{a,1}H]$ has a basis consisting of elements $c \cdot Sq^I \operatorname{red}_2$ with $c \in H^{p,b}(\mathbf{Q}, \mathbf{Z}/2)$ and either $I = \varnothing$ or $I = (i_1, \dots, i_t)$ admissible with $i_t \geq 2$. When $I = \varnothing$, we have $(p, b) = (a, 1)$, and $H^{a,1}(\mathbf{Q}, \mathbf{Z}/2) = 0$ for $a \geq 2$ [14, Theorem 3.6]. When $I \neq \varnothing$ with $i_t \geq 2$, the monomial Sq^I has positive weight and degree at least 2; since $H^{p,b}(\mathbf{Q}, \mathbf{Z}/2) = 0$ for $b < 0$ and $H^{p,0}(\mathbf{Q}, \mathbf{Z}/2)$ equals $\mathbf{F}_2$ for $p = 0$ and 0 otherwise, only $(p, b) = (0, 0)$ and weight-1 monomials contribute. The sole admissible monomials of weight 1 ending in an index at least 2 are Sq^2 in bidegree $(2, 1)$ and Sq^3 in bidegree $(3, 1)$. Thus $[H\mathbf{Z}_{(2)}, \Sigma^{1,1}H]$ has zero reduced summand, $[H\mathbf{Z}_{(2)}, \Sigma^{2,1}H] = \mathbf{F}_2 \cdot (Sq^2 \operatorname{red}_2)$, and $[H\mathbf{Z}_{(2)}, \Sigma^{3,1}H] = \mathbf{F}_2 \cdot (Sq^3 \operatorname{red}_2)$. Finally, since $H^{2,1}(\mathbf{Q}, \mathbf{Z}_{(2)}) = 0$, every $x \in [H\mathbf{Z}_{(2)}, \Sigma^{2,1}H\mathbf{Z}_{(2)}]$ satisfies $2x = 0$ by Lemma 5.1 and therefore lifts along ∂_2 to the reduced summand of $[H\mathbf{Z}_{(2)}, \Sigma^{1,1}H]$, which is zero.

(2) In (5.1) for $s = 1$, part (1) and Lemma 2.1 show that $\beta^*\colon [\mathrm{kgl}_{(2)}, \Sigma^{2,1}H\mathbf{Z}_{(2)}] \to \mathbf{Z}_{(2)}u$ is injective. By Theorem B(1), $\beta^*(c_3^{\mathrm{st}}) = \pm 2u$. Let $f\colon \mathbf{P}^2 \to BGL_1 \to BGL$ classify the line bundle $\mathcal{O}(1)$ trivialized at a rational basepoint, and put $t = c_1(\mathcal{O}(1)) \in \widetilde{H}^{2,1}(\mathbf{P}^2, \mathbf{Z})$. By Proposition 3.7(1), $f^*\Omega^\infty(\Sigma^{2,1}c_3^{\mathrm{st}}) = \pm t^2$, which generates $\widetilde{H}^{4,2}(\mathbf{P}^2, \mathbf{Z}_{(2)}) = \mathbf{Z}_{(2)}t^2$ and reduces to $\bar{t}^2 \neq 0$ in $\widetilde{H}^{4,2}(\mathbf{P}^2, \mathbf{Z}/2)$ [14, Theorem 15.12]. In particular, c_3^{st} is not divisible by 2, so $u \notin \operatorname{im}(\beta^*)$ and $\Delta = \delta_0^*(u) = (\Sigma^{3,1}u) \circ \delta_0$ is nonzero in $[H\mathbf{Z}_{(2)}, \Sigma^{3,1}H\mathbf{Z}_{(2)}]$. Since $2 \cdot [H\mathbf{Z}_{(2)}, \Sigma^{3,1}H\mathbf{Z}_{(2)}] = 0$ (Lemma 5.1), postcomposition with red_2 is injective on $[H\mathbf{Z}_{(2)}, \Sigma^{3,1}H\mathbf{Z}_{(2)}]$, so part (1) gives $\operatorname{red}_2 \circ \Delta = Sq^3 \circ \operatorname{red}_2$.

Since $d_1\colon s_w\mathrm{KGL}_{(2)} \to \Sigma^{1,0}s_{w+1}\mathrm{KGL}_{(2)}$ is $\Sigma^{2w,w}\Delta$, this proves (3). Applying $[-, \Sigma^{a,1}H]$ to (2.1) and using Lemma 2.1 gives exact sequences

$$\mathbf{F}_2 \cdot \Sigma^{3,1}(\operatorname{red}_2 u) \xrightarrow{\delta_0^*} [H\mathbf{Z}_{(2)}, \Sigma^{3,1}H] \xrightarrow{u^*} [\mathrm{kgl}_{(2)}, \Sigma^{3,1}H] \longrightarrow 0$$

and

$$[H\mathbf{Z}_{(2)}, \Sigma^{2,1}H] \xrightarrow{u^*} [\mathrm{kgl}_{(2)}, \Sigma^{2,1}H] \xrightarrow{\beta^*} \mathbf{F}_2 \cdot (\operatorname{red}_2 u) \xrightarrow{\delta_0^*} [H\mathbf{Z}_{(2)}, \Sigma^{3,1}H].$$

Because δ_0^* sends the generator to $\operatorname{red}_2 \circ \Delta = Sq^3 \operatorname{red}_2$, it is an isomorphism. The first sequence gives $[\mathrm{kgl}_{(2)}, \Sigma^{3,1}H] = 0$ and $Q_1 \operatorname{red}_2 u = Sq^3 \operatorname{red}_2 u = 0$, proving (2). The second sequence shows that $[\mathrm{kgl}_{(2)}, \Sigma^{2,1}H]$ is spanned by $Sq^2 \operatorname{red}_2 u$; since $f^*\Omega^\infty(\Sigma^{2,1} \operatorname{red}_2 c_3^{\mathrm{st}}) = \bar{t}^2 \neq 0$, we have $\operatorname{red}_2 c_3^{\mathrm{st}} \neq 0$ and therefore $\operatorname{red}_2 c_3^{\mathrm{st}} = Sq^2 \operatorname{red}_2 u$, proving (4). □

Proposition 5.3. *Over* $\mathbf{Q}$*:*

(1) There exist morphisms $\theta_s\colon \mathrm{kgl}_{(2)} \to \Sigma^{2s,s}H\mathbf{Z}_{(2)}$ *for* $s \geq 0$ *with* $\theta_0 = u$, $\theta_1 = \pm c_3^{\mathrm{st}}$, $\operatorname{red}_2 \theta_1 = Sq^2 \operatorname{red}_2 u$, *and*

$$\beta^*\theta_s = 2\theta_{s-1}, \qquad \textit{Bott value of } \theta_s = 2^s \qquad (s \geq 1).$$

(2) For every $r \geq 2$*, there exists* $\kappa^{(r)}\colon \mathrm{kgl}_{(2)} \to \Sigma^{2r,r}H\mathbf{Z}_{(2)}$ *with* $2\kappa^{(r)} = 0$ *and* $\beta^*\kappa^{(r)} = 0$ *such that for both* $n = 2r + 1$ *and* $n = 2r + 2$*, the* 2*-localization of* c_n^{st} *satisfies*

$$c_n^{\mathrm{st}} = \pm \frac{(n-1)!}{2^r}\, \theta_r + \kappa^{(r)}, \tag{5.3}$$

and $v_2((n-1)!/2^r) = v_2(r!) \geq 1$.

Proof.

(1) Set $\theta_0 = u$, which has Bott value 1. By Theorem B(1) and Lemma 2.1, $\beta^*(c_3^{\mathrm{st}}) = \pm 2u$; choosing the sign of $\theta_1 = \pm c_3^{\mathrm{st}}$ gives $\beta^*\theta_1 = 2\theta_0$, Bott value 2, and $\mathrm{red}_2\,\theta_1 = Sq^2\,\mathrm{red}_2\,u$ (Lemma 5.2(4)). Given θ_{s-1} for $s \geq 2$, the target of δ_0^* in (5.1) is killed by 2 (Lemma 5.1), so $\delta_0^*(2\theta_{s-1}) = 0$. Exactness of (5.1) therefore gives $2\theta_{s-1} = \beta^*\theta_s$ for a morphism $\theta_s\colon \mathrm{kgl}_{(2)} \to \Sigma^{2s,s}H\mathbf{Z}_{(2)}$ of Bott value 2^s.

(2) We prove by induction on $r \geq 1$ that $c_{2r+1}^{\mathrm{st}} = \pm\frac{(2r)!}{2^r}\theta_r + \kappa^{(r)}$ with $2\kappa^{(r)} = 0$ and $\beta^*\kappa^{(r)} = 0$, setting $\kappa^{(1)} = 0$ for $r = 1$. For $r \geq 2$, the induction hypothesis, Proposition 3.7(3), and the fact that $2r\kappa^{(r-1)} = 0$ give

$$\beta^*(c_{2r+1}^{\mathrm{st}}) = \pm 2r(2r-1)\Big(\pm\frac{(2r-2)!}{2^{r-1}}\,\theta_{r-1} + \kappa^{(r-1)}\Big) = \epsilon_r\frac{(2r)!}{2^{r-1}}\,\theta_{r-1} = \beta^*\Big(\epsilon_r\frac{(2r)!}{2^r}\,\theta_r\Big)$$

for a sign $\epsilon_r \in \{\pm 1\}$. Defining $\kappa^{(r)} = c_{2r+1}^{\mathrm{st}} - \epsilon_r\frac{(2r)!}{2^r}\theta_r$ gives $\beta^*\kappa^{(r)} = 0$; exactness of (5.1) lifts $\kappa^{(r)}$ to $[H\mathbf{Z}_{(2)}, \Sigma^{2r,r}H\mathbf{Z}_{(2)}]$, so $2\kappa^{(r)} = 0$ by Lemma 5.1. For $n = 2r+2$, Proposition 3.7(2) gives $c_{2r+2}^{\mathrm{st}} = \pm(2r+1)c_{2r+1}^{\mathrm{st}}$; since $2r+1$ is odd and $2\kappa^{(r)} = 0$, this gives (5.3) with the same $\kappa^{(r)}$. Finally, for $n \in \{2r+1, 2r+2\}$ with $r \geq 2$, we have $v_2((n-1)!) = v_2((2r)!) = r + v_2(r!)$ with $v_2(r!) \geq 1$. □

5.3. **Mod-2 motivic cohomology and the cases $n = 3, 4$.** Let F be a field of characteristic zero. By the norm residue and Beilinson–Lichtenbaum theorems [20],

$$H^{*,*}(F, \mathbf{Z}/2) = (K_*^M(F)/2)[\tau], \qquad \tau \in H^{0,1}(F, \mathbf{Z}/2),$$

where τ corresponds to $-1 \in \mu_2(F)$. Set $\rho = \{-1\} \in K_1^M(F)$ and write $\bar\rho \in K_1^M(F)/2$ for its mod-2 reduction. Under $\mathbf{Z}(1) \simeq \mathbf{G}_m[-1]$ [14, Theorem 4.1], we have $\partial_2\tau = \rho$ and $Sq^1\tau = \bar\rho$. Positive Steenrod squares vanish on $K_*^M(F)/2$ because the degree exceeds the weight, while $Sq^2\tau = 0$ [19, Lemma 9.9]. For $k > 2$, the class $Sq^k\tau$ lies in $H^{k,1+\lfloor k/2\rfloor}(F, \mathbf{Z}/2)$, which vanishes because $k > 1 + \lfloor k/2\rfloor$ [14, Theorem 3.6]. Because Voevodsky's Cartan formula [19, Proposition 9.7] has coefficients in $\mathbf{F}_2[\tau, \bar\rho]$, the subring $\mathbf{F}_2[\tau, \bar\rho]$ is closed under every admissible monomial Sq^I, and $Sq^I(\tau^h\bar A) = Sq^I(\tau^h)\,\bar A$ for all $\bar A \in K_*^M(F)/2$ and $h \geq 1$. In low degrees, the first two identities below hold for $h \geq 1$ and the third for $h \geq 2$:

$$Sq^1(\tau^h) = h\tau^{h-1}\bar\rho, \qquad Sq^2(\tau^h) = \tbinom{h}{2}\tau^{h-1}\bar\rho^2, \qquad Sq^3(\tau^h) = (h-1)\tbinom{h}{2}\tau^{h-2}\bar\rho^3. \tag{5.4}$$

For $h = 1$, we have $Sq^3\tau = 0$. On mod-2 reductions of integral classes, Sq^1 vanishes and Q_1 acts as $Sq^1Sq^2 = Sq^3$.

Proof of Theorem D for $n = 3, 4$. Here $r = 1$, $m = j+1$, and $v_2((n-1)!) = 1$. By Proposition 3.7(2) and Proposition 5.3(1), we have $c_3^{\mathrm{st}} = \pm\theta_1$ and $c_4^{\mathrm{st}} = \pm 3\theta_1$. For any cover class $y \in \pi_{-j,-m}\mathrm{kgl}(F)$, we have $\mathrm{red}_2\,u(y) \in H^{j,j+1}(F, \mathbf{Z}/2) = \tau\cdot(K_j^M(F)/2)$, so $\mathrm{red}_2\,u(y) = \tau\bar A$ for some $\bar A \in K_j^M(F)/2$. By Proposition 5.3(1),

$$\mathrm{red}_2\,\theta_1(y) = Sq^2(\tau\bar A) = Sq^2(\tau)\bar A = 0,$$

so $\theta_1(y) \in 2K_n^M(F)_{(2)}$, and the image of $c_n^{\mathrm{st}}(y)$ in $K_n^M(F)_{(2)}$ lies in $\{\pm\theta_1(y), \pm 3\theta_1(y)\} \subseteq 2K_n^M(F)_{(2)}$. □

5.4. **Vanishing of the torsion summand $\kappa^{(r)}$.**

Lemma 5.4. *Let $r \geq 2$ and let $\kappa^{(r)}$ be as in (5.3). There exists $\epsilon_r \in \mathbf{F}_2$ such that for every $j \in \{1, 2\}$ (with $n = 2r + j$ and $m = r + j$), every field F of characteristic zero, and every cover class $y \in \pi_{-j,-m}\mathrm{kgl}(F)$, if $\bar A \in K_j^M(F)/2$ is the unique class with $\mathrm{red}_2\,u(y) = \tau^r\bar A$ in $H^{j,m}(F, \mathbf{Z}/2)$ and $A \in K_j^M(F)$ is an integral lift of $\bar A$, then $\kappa^{(r)}(y) = \epsilon_r\rho^{2r}A$ in $K_n^M(F)_{(2)}$.*

Proof. Since $\beta^*\kappa^{(r)} = 0$, exactness of (5.1) gives $\kappa^{(r)} = \widetilde{\kappa}\circ u$ for some $\widetilde{\kappa} \in [H\mathbf{Z}_{(2)}, \Sigma^{2r,r}H\mathbf{Z}_{(2)}]$ over $\mathbf{Q}$, and Lemma 5.1 gives $2\widetilde{\kappa} = 0$. Thus $\widetilde{\kappa}$ factors through $\mathrm{red}_2\colon H\mathbf{Z}_{(2)} \to H$ as $\varphi \circ \mathrm{red}_2$ for some $\varphi\colon H \to \Sigma^{2r,r}H\mathbf{Z}_{(2)}$. Since $2\cdot \mathrm{id}_H = 0$, we also have $2\varphi = 0$, so φ lifts along $\partial_2\colon \Sigma^{2r-1,r}H \to \Sigma^{2r,r}H\mathbf{Z}_{(2)}$ to an operation $\Psi \in \mathcal{A}$ of bidegree $(2r-1, r)$ with

$$\kappa^{(r)} = \partial_2 \circ \Psi \circ \mathrm{red}_2 \circ u.$$

Write Ψ as a sum of terms $\tau^b z\, Sq^I$ with $z \in K^M_a(\mathbf{Q})/2$ and Sq^I admissible of bidegree (p, w), so $a + p = 2r - 1$ and $a + b + w = r$. For each term, $Sq^I(\tau^r\bar{A}) = Sq^I(\tau^r)\bar{A}$, where $Sq^I(\tau^r) \in \mathbf{F}_2[\tau, \bar{\rho}]$ has bidegree $(p, r+w)$ and is therefore an $\mathbf{F}_2$-multiple of $\tau^{r+w-p}\bar{\rho}^p$. Since $b + r + w - p = 2r - a - p = 1$, each term $\tau^b z\, Sq^I(\tau^r\bar{A})$ is an $\mathbf{F}_2$-multiple of $\tau z\bar{\rho}^p\bar{A}$, and summing over all terms gives $\Psi(\tau^r\bar{A}) = \tau D\bar{A}$ for a class $D \in K^M_{2r-1}(\mathbf{Q})/2$ depending only on Ψ.

Because $2r - 1 \geq 3$, Tate's theorem [15, Theorem A.2] gives an isomorphism

$$K^M_{2r-1}(\mathbf{Q})/2 \xrightarrow{\sim} K^M_{2r-1}(\mathbf{R})/2 = \mathbf{F}_2 \cdot \bar{\rho}^{2r-1},$$

so $D = \epsilon_r\bar{\rho}^{2r-1}$ for some $\epsilon_r \in \mathbf{F}_2$. By $H\mathbf{Z}_{(2)}$-linearity of ∂_2, we have $\partial_2(\tau\cdot\mathrm{red}_2\, c) = (\partial_2\tau)c = \rho c$ for integral classes c, whence $\kappa^{(r)}(y) = \partial_2(\epsilon_r\tau\cdot\mathrm{red}_2(\rho^{2r-1}A)) = \epsilon_r\rho^{2r}A$. □

Lemma 5.5. *For every $a \geq 2$ and $w \geq 0$, there is a direct sum decomposition*

$$H^{a,w}(\mathbf{R}, \mathbf{Z}_{(2)}) = V_{a,w} \oplus T_{a,w},$$

where $V_{a,w}$ is uniquely 2-divisible and

$$T_{a,w} = H^{a,w}(\mathbf{R}, \mathbf{Z}_{(2)})[2] = \begin{cases} \mathbf{Z}/2\cdot g_{a,w}, & w \geq a \text{ and } w-a \text{ is even}, \\ 0, & \text{otherwise}, \end{cases}$$

with $g_{a,w} = \partial_2(\tau^{w-a+1}\bar{\rho}^{a-1})$ satisfying $\mathrm{red}_2(g_{a,w}) = \tau^{w-a}\bar{\rho}^a \in H^{a,w}(\mathbf{R}, \mathbf{Z}/2)$.

Proof. For $w < a$, we have $H^{a,w}(\mathbf{R}, \mathbf{Z}_{(2)}) = 0$ [14, Theorem 3.6]. Assume $w \geq a \geq 2$. Over $\mathbf{C} = \mathbf{R}(\sqrt{-1})$, the Beilinson–Lichtenbaum theorem gives $H^{b,w}(\mathbf{C}, \mathbf{Z}/2) \cong H^b_{\mathrm{et}}(\mathbf{C}, \mu_2^{\otimes w}) = 0$ for $1 \leq b \leq w$. Because $1 \leq a - 1 < a \leq w$, both $H^{a-1,w}(\mathbf{C}, \mathbf{Z}/2)$ and $H^{a,w}(\mathbf{C}, \mathbf{Z}/2)$ vanish, so multiplication by 2 on $H^{a,w}(\mathbf{C}, \mathbf{Z}_{(2)})$ is bijective. Let $B = H^{a,w}(\mathbf{C}, \mathbf{Z}_{(2)})$ and let σ generate $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$. Since $N_{\mathbf{C}/\mathbf{R}} \circ \mathrm{res}_{\mathbf{C}/\mathbf{R}} = 2$ and $\mathrm{res}_{\mathbf{C}/\mathbf{R}} \circ N_{\mathbf{C}/\mathbf{R}} = 1 + \sigma$, the map $s\colon B^\sigma \to H^{a,w}(\mathbf{R}, \mathbf{Z}_{(2)})$ defined by $s(b) = N_{\mathbf{C}/\mathbf{R}}(b/2)$ is a section of restriction. Here $b/2$ is formed using unique division by 2 in B. Setting $V_{a,w} = \mathrm{im}(s)$ and $T_{a,w} = \ker(\mathrm{res}_{\mathbf{C}/\mathbf{R}})$ gives $H^{a,w}(\mathbf{R}, \mathbf{Z}_{(2)}) = V_{a,w} \oplus T_{a,w}$. The group $V_{a,w} \cong B^\sigma$ is uniquely 2-divisible, while $2T_{a,w} = 0$. Since B has no 2-torsion, this kernel is exactly $H^{a,w}(\mathbf{R}, \mathbf{Z}_{(2)})[2]$.

Since $K^M_*(\mathbf{R})/2 = \mathbf{F}_2[\bar{\rho}]$ [15, Example 1.6], we have $H^{*,*}(\mathbf{R}, \mathbf{Z}/2) = \mathbf{F}_2[\tau, \bar{\rho}]$, so each group $H^{b,w}(\mathbf{R}, \mathbf{Z}/2) = \mathbf{F}_2 \cdot \tau^{w-b}\bar{\rho}^b$ $(1 \leq b \leq w)$ is 1-dimensional over $\mathbf{F}_2$. Because multiplication by 2 is bijective on $V_{a,w}$ and $V_{a+1,w}$ and zero on $T_{a,w}$ and $T_{a+1,w}$, the coefficient sequence reduces to a short exact sequence

$$0 \longrightarrow T_{a,w} \xrightarrow{\mathrm{red}_2} H^{a,w}(\mathbf{R}, \mathbf{Z}/2) \xrightarrow{\partial_2} T_{a+1,w} \longrightarrow 0.$$

If $w - a$ is even, $g_{a,w} = \partial_2(\tau^{w-a+1}\bar{\rho}^{a-1}) \in T_{a,w}$ satisfies

$$\mathrm{red}_2(g_{a,w}) = Sq^1(\tau^{w-a+1}\bar{\rho}^{a-1}) = \tau^{w-a}\bar{\rho}^a \neq 0,$$

so $\mathrm{red}_2\colon T_{a,w} \xrightarrow{\sim} H^{a,w}(\mathbf{R}, \mathbf{Z}/2)$ is an isomorphism. If $w - a$ is odd, then $w \geq a + 1$ and $w - (a + 1)$ is even, so $\partial_2(\tau^{w-a}\bar{\rho}^a) = g_{a+1,w}$ generates $T_{a+1,w} \cong \mathbf{F}_2$; hence $\partial_2\colon H^{a,w}(\mathbf{R}, \mathbf{Z}/2) \xrightarrow{\sim} T_{a+1,w}$ is an isomorphism and $T_{a,w} = 0$. □

Lemma 5.6. *Let $r \geq 4$ be a multiple of 4, and put $n = 2r + 1$ and $m = r + 1$. There exists a cover class $x_0 \in \pi_{-1,-m}\mathrm{kgl}(\mathbf{R})$ with $\mathrm{red}_2\, u(x_0) = \tau^r\bar{\rho}$.*

Proof. Put $\xi = \partial_2(\tau^{r+1}) \in H^{1,m}(\mathbf{R}, \mathbf{Z}_{(2)})$, so $2\xi = 0$ and $\mathrm{red}_2(\xi) = (r + 1)\tau^r\bar{\rho} = \tau^r\bar{\rho}$. Any preimage of ξ under the edge map $\pi_{n,0}(f_m\mathrm{KGL}_{(2)})(\mathbf{R}) \to H^{1,m}(\mathbf{R}, \mathbf{Z}_{(2)})$ has the form x_0/b

with $x_0 \in \pi_{n,0}(f_m\mathrm{KGL})(\mathbf{R}) = \pi_{-1,-m}\mathrm{kgl}(\mathbf{R})$ and b odd; since $2\xi = 0$, such an x_0 satisfies $u(x_0) = b\xi = \xi$ in $H^{1,m}(\mathbf{R}, \mathbf{Z}_{(2)})$ and $\mathrm{red}_2\, u(x_0) = \tau^r\bar{\rho}$.

Since $f_{n+1}\mathrm{KGL}_{(2)}$ vanishes on $\pi_{n,0}$ and $\pi_{n-1,0}$ by (2.2), the quotient map $f_m\mathrm{KGL}_{(2)} \to f_m\mathrm{KGL}_{(2)}/f_{n+1}\mathrm{KGL}_{(2)}$ induces an isomorphism on $\pi_{n,0}$. Filtering $f_m\mathrm{KGL}_{(2)}/f_{n+1}\mathrm{KGL}_{(2)}$ over $\mathbf{R}$ by the images of $f_w\mathrm{KGL}_{(2)}/f_{n+1}\mathrm{KGL}_{(2)}$ ($m \le w \le n+1$) gives a spectral sequence supported in weights $m \le w \le n$, with $E_1^{a,w} = \pi_{2w-a,0}(s_w\mathrm{KGL}_{(2)})(\mathbf{R}) = H^{a,w}(\mathbf{R}, \mathbf{Z}_{(2)})$ in total stem $v = 2w - a$ and differentials $d_t\colon E_t^{a,w} \to E_t^{a+2t+1,w+t}$. Because $\xi \in E_1^{1,m}$ lies at the bottom weight m, no differential enters $(1, m)$, so ξ lifts to $\pi_{n,0}(f_m\mathrm{KGL}_{(2)})(\mathbf{R})$ if and only if $d_t(\xi) = 0$ for all $t \ge 1$.

The Adams operation Ψ^3 acts on $s_w\mathrm{KGL}_{(2)} \simeq \Sigma^{2w,w}H\mathbf{Z}_{(2)}$ as 3^w, so $3^w(3^t - 1)$ and hence $2^{v_2(3^t-1)}$ kills $\mathrm{im}(d_t)$ out of weight w for every $t \ge 1$. By Lemma 5.2(3), $\mathrm{red}_2 \circ d_1 = Sq^3 \circ \mathrm{red}_2$ on E_1.

(1) For $t = 1$, we have $2d_1(\xi) = 0$, so $d_1(\xi) \in T_{4,m+1}$. Since $4 \mid r$, formula (5.4) gives
$$\mathrm{red}_2(d_1(\xi)) = Sq^3(\tau^r\bar{\rho}) = (r-1)\tbinom{r}{2}\tau^{r-2}\bar{\rho}^4 = 0,$$
and injectivity of red_2 on $T_{4,m+1}$ (Lemma 5.5) gives $d_1(\xi) = 0$.

(2) Fix $t \ge 2$ and put $(a, w) = (2t+2, m+t)$, so $a \ge 6$ and $w - a = r - t - 1$. If $w < a$ (that is, $t \ge r$), then $E_1^{a,w} = 0$. Suppose $w \ge a$, so $2 \le t \le r-1$ and $m + 2 \le w \le n - 1$; then both $(a-3, w-1)$ and $(a+3, w+1)$ have degree at least 3 and weight in $[m, n]$. In the splitting $E_1^{b,v} = V_{b,v} \oplus T_{b,v}$ of Lemma 5.5, $d_1(V_{b,v})$ is both 2-divisible and killed by 2, hence 0, while $d_1(T_{b,v}) \subseteq T_{b+3,v+1}$. For even $k = v - b \ge 0$, formula (5.4) gives $\mathrm{red}_2\, d_1(g_{b,v}) = (k-1)\binom{k}{2}\tau^{k-2}\bar{\rho}^{b+3}$, so $d_1(g_{b,v}) = g_{b+3,v+1}$ when $k \equiv 2 \pmod 4$ and $d_1(g_{b,v}) = 0$ when $k \equiv 0 \pmod 4$. At (a, w): if $w - a$ is odd, then $T_{a,w} = 0$; if $w - a \equiv 2 \pmod 4$, then $d_1(g_{a,w}) = g_{a+3,w+1} \neq 0$ in $T_{a+3,w+1}$; if $w - a \equiv 0 \pmod 4$, then $(w-1) - (a-3) = w - a + 2 \equiv 2 \pmod 4$ and $d_1(g_{a-3,w-1}) = g_{a,w}$. In every case the T-summands are exact at (a, w), so $E_2^{a,w} = V_{a,w}$.

By induction on $s \in \{2, \dots, t\}$, every differential d_s into or out of the uniquely 2-divisible group $E_s^{a,w} = V_{a,w}$ has image of bounded 2-exponent and therefore vanishes, so $E_t^{a,w} = V_{a,w}$ is torsion-free. Since $2\xi = 0$, we obtain $d_t(\xi) = 0$. □

Lemma 5.7. *Put $\bar{c}_w = \mathrm{red}_2\, c_w \in H^{2w,w}(BGL, \mathbf{Z}/2)$. For every $w \ge 1$,*
$$Sq^2(\bar{c}_w) = \bar{c}_1\bar{c}_w + (w+1)\bar{c}_{w+1} \qquad \text{in } H^{2w+2,w+1}(BGL, \mathbf{Z}/2).$$

Proof. Choose $N \ge w + 2$. The universal Chow ring calculation [7, §6.1] identifies the diagonal cohomology of BGL and BGL_N with the polynomial rings in $\bar{c}_1, \bar{c}_2, \dots$ and $\bar{c}_1, \dots, \bar{c}_N$, respectively. Hence restriction $H^{2w+2,w+1}(BGL, \mathbf{Z}/2) \to H^{2w+2,w+1}(BGL_N, \mathbf{Z}/2)$ is an isomorphism in the indicated weight. By the projective bundle theorem on the complete flag bundle [14, Theorem 15.12], $H^{*,*}(BGL_N, \mathbf{Z}/2)$ injects into $H^{*,*}(BGL_1^N, \mathbf{Z}/2)$, sending $\bar{c}_1, \dots, \bar{c}_N$ to the elementary symmetric polynomials $\bar{e}_1, \dots, \bar{e}_N$ in the Chern roots $x_1, \dots, x_N \in H^{2,1}(BGL_1^N, \mathbf{Z}/2)$. Each x_α is the reduction of an integral Chern class, so $Sq^1(x_\alpha) = 0$; also $Sq^2(x_\alpha) = x_\alpha^2$ [19, Lemma 9.8]. Therefore, Voevodsky's Cartan formula [19, Proposition 9.7] shows that Sq^2 acts on $\mathbf{F}_2[x_1, \dots, x_N]$ as the derivation sending x_α to x_α^2. Reducing the polynomial identity
$$e_1 e_w = \sum_{1 \le \alpha_1 < \cdots < \alpha_w \le N} \sum_{s=1}^{w} x_{\alpha_1} \cdots x_{\alpha_s}^2 \cdots x_{\alpha_w} + (w+1)e_{w+1}$$
modulo 2 gives $Sq^2(\bar{e}_w) = \bar{e}_1\bar{e}_w + (w+1)\bar{e}_{w+1}$, which pulls back to the stated identity. □

Proposition 5.8. *Let $n \ge 5$ with m, r, j as in (1.1). For every field F of characteristic zero and every cover class $y \in \pi_{-j,-m}\mathrm{kgl}(F)$, we have $\kappa^{(r)}(y) = 0$ in $K_n^M(F)_{(2)}$, and therefore*
$$c_n^{\mathrm{st}}(y) = \pm\frac{(n-1)!}{2^r}\,\theta_r(y) \qquad \text{in } K_n^M(F)_{(2)}. \tag{5.5}$$

Proof. Let $\epsilon_r \in \mathbf{F}_2$, $\bar{A} \in K_j^M(F)/2$, and $A \in K_j^M(F)$ be as in Lemma 5.4.

(1) Suppose $4 \nmid r$. If r is odd, then $0 = Sq^1 \operatorname{red}_2 u(y) = Sq^1(\tau^r)\bar{A} = \tau^{r-1}\bar{\rho}\bar{A}$ by (5.4), so $\bar{\rho}\bar{A} = 0$. If $r \equiv 2 \pmod 4$, then $0 = Q_1 \operatorname{red}_2 u(y) = (r-1)\binom{r}{2}\tau^{r-2}\bar{\rho}^3\bar{A} = \tau^{r-2}\bar{\rho}^3\bar{A}$ by Lemma 5.2(2) and (5.4), so $\bar{\rho}^3\bar{A} = 0$. In either case, $\rho^k A \in 2K^M_{j+k}(F)$ for some $k \in \{1,3\}$ with $k < 2r$; since $2\rho = 0$, we obtain $\rho^{2r}A = 0$ and $\kappa^{(r)}(y) = 0$.

(2) Suppose $4 \mid r$, put $n_0 = 2r+1$, and let $x_0 \in \pi_{-1,-(r+1)}\mathrm{kgl}(\mathbf{R})$ be the cover class of Lemma 5.6 with image $\bar{x}_0 \in K_{n_0}(\mathbf{R}) = [S^{n_0,0}, BGL]_{\mathcal{H}_*(\mathbf{R})}$. Write $c_{n_0,w}(\bar{x}_0) \in H^{2w-n_0,w}(\mathbf{R},\mathbf{Z})$ for the pullback of c_w along $\bar{x}_0 \colon S^{n_0,0} \to BGL$. Since $S^{n_0,0}$ is a simplicial suspension ($n_0 \geq 9$), cup products vanish in $\widetilde{H}^{*,*}(S^{n_0,0},\mathbf{Z}/2)$, so Lemma 5.7 with $w = 2r$ gives

$$Sq^2 \operatorname{red}_2 c_{n_0,2r}(\bar{x}_0) = (2r+1)\operatorname{red}_2 c_{n_0,n_0}(\bar{x}_0) = \operatorname{red}_2 c^{\mathrm{st}}_{n_0}(x_0).$$

Because $\operatorname{red}_2 c_{n_0,2r}(\bar{x}_0) \in H^{2r-1,2r}(\mathbf{R},\mathbf{Z}/2) = \mathbf{F}_2 \cdot \tau\bar{\rho}^{2r-1}$ is killed by Sq^2, we have $\operatorname{red}_2 c^{\mathrm{st}}_{n_0}(x_0) = 0$. On the other hand, since $(2r)!/2^r$ is even (Proposition 5.3(2)), formula (5.3) and Lemma 5.4 give

$$0 = \operatorname{red}_2 c^{\mathrm{st}}_{n_0}(x_0) = \operatorname{red}_2 \kappa^{(r)}(x_0) = \epsilon_r \bar{\rho}^{n_0} \qquad \text{in } K^M_{n_0}(\mathbf{R})/2 = \mathbf{F}_2 \cdot \bar{\rho}^{n_0}.$$

Thus $\epsilon_r = 0$, so $\kappa^{(r)}(y) = 0$ for every $j \in \{1,2\}$, F, and y. □

6. Quadratic descent at the prime 2

6.1. **Kahn's decomposition over fields containing $\sqrt{-1}$.** Let L be a field of characteristic zero containing $\sqrt{-1}$, and let $q = 2^\nu$ with $\nu \geq 3$. Because the 2-adic cyclotomic character of L takes values in $1 + 4\mathbf{Z}_2$, Kahn's theorem [12, Theorem 1(a) and Proposition 2.1] gives an isomorphism, natural with respect to field homomorphisms,

$$B^n_L \colon \bigoplus_{w=0}^{n} H^{2w-n}_{\mathrm{et}}(L, \mu_q^{\otimes w}) \xrightarrow{\sim} K_n(L;\mathbf{Z}/q)$$

for every $n \geq 1$. Its weight-n component is the canonical map $K^M_n(L)/q \to K_n(L;\mathbf{Z}/q)$, and the components with $2w-n < 0$ vanish. For $\lceil n/2 \rceil \leq w < n$, its weight-$w$ component is spanned by transfers $\operatorname{Tr}_{L'/L}(\mathfrak{b} \cdot \iota_{2w-n}(A))$, where L'/L splits $\mu_q^{\otimes(n-w)}$, $\mathfrak{b} \in K_{2(n-w)}(L';\mathbf{Z}/q)$ is a Bott element, and $A \in K^M_{2w-n}(L')$ [12, (2.3)]; in degree zero, $\iota_0 \colon \mathbf{Z} \to K_0(L')$ sends 1 to the class of the trivial line bundle. Define the Milnor projection by

$$p_L = \mathrm{pr}_{w=n} \circ (B^n_L)^{-1} \colon K_n(L;\mathbf{Z}/q) \longrightarrow K^M_n(L)/q.$$

Lemma 6.1. *Let L be a field of characteristic zero containing $\sqrt{-1}$, let $n \geq 3$ with m, r, j as in (1.1), and let $q = 2^\nu$ with $\nu \geq 3$. The canonical map $\pi_{-j,-m}(\mathrm{kgl}/q)(L) \to K_n(L;\mathbf{Z}/q)$ is injective, and its image equals the direct sum of the components of B^n_L of weight $w \geq m$.*

Proof. The cofiber of $f_m\mathrm{KGL}/q \to \mathrm{kgl}/q$ is an iterated extension of $\Sigma^{2w,w}H\mathbf{Z}/q$ for $0 \leq w < m$, whose $\pi_{n+1,0}$ equals $H^{2w-n-1,w}(L,\mathbf{Z}/q) = 0$ since $2w - n - 1 < 0$. Hence $\pi_{-j,-m}(\mathrm{kgl}/q)(L) \to \pi_{n,0}(\mathrm{kgl}/q)(L) \cong K_n(L;\mathbf{Z}/q)$ is injective.

For $w \geq m$, put $t = n - w \in \{0,\dots,r\}$. Given a generator $\operatorname{Tr}_{L'/L}(\mathfrak{b} \cdot \iota_{2w-n}(A))$ of weight w, Lemma 2.5 lifts $\mathfrak{b}$ uniquely to $\pi_{0,-t}(\mathrm{kgl}/q)(L')$, and for any sphere lift $\widehat{A}$ of A, Lemma 2.7 shows that $\beta^{w-m}\operatorname{Tr}_{L'/L}(\mathfrak{b} \cdot \widehat{A}) \in \pi_{-j,-m}(\mathrm{kgl}/q)(L)$ maps to $\operatorname{Tr}_{L'/L}(\mathfrak{b} \cdot \iota_{2w-n}(A))$. Thus all components of weight $w \geq m$ lie in the image.

For $w < m$, we have $2w - n < 0$ unless $n = 2s$ is even and $w = s = m-1$, where the sole remaining component is $H^0_{\mathrm{et}}(L, \mu_q^{\otimes s})$. In that case $m = s+1$, and Lemma 2.5 identifies $\pi_{2s,0}(f_s\mathrm{KGL}/q)(L)$ with $K_{2s}(L;\mathbf{Z}/q)$. The cofiber sequence $f_{s+1}\mathrm{KGL}/q \to f_s\mathrm{KGL}/q \xrightarrow{u} \Sigma^{2s,s}H\mathbf{Z}/q$ therefore identifies the image of $\pi_{2s,0}(f_m\mathrm{KGL}/q)(L)$ in $K_{2s}(L;\mathbf{Z}/q)$ with the

kernel of $u\colon K_{2s}(L;\mathbf{Z}/q) \to H^{0,s}(L,\mathbf{Z}/q)$. Over an algebraic closure $\bar{L}$, all components of $B^{2s}_{\bar{L}}$ of weight $w > s$ vanish since $\mathrm{cd}_2(\bar{L}) = 0$, so the commutative diagram

$$\begin{array}{ccccc} H^0_{\mathrm{et}}(L,\mu_q^{\otimes s}) & \xrightarrow{B^{2s}_{L,w=s}} & K_{2s}(L;\mathbf{Z}/q) & \xrightarrow{u} & H^{0,s}(L,\mathbf{Z}/q) \\ \downarrow & & \downarrow & & \downarrow \\ H^0_{\mathrm{et}}(\bar{L},\mu_q^{\otimes s}) & \xrightarrow[\sim]{} & K_{2s}(\bar{L};\mathbf{Z}/q) & \xrightarrow[\sim]{u} & H^{0,s}(\bar{L},\mathbf{Z}/q) \end{array}$$

has bottom arrows isomorphisms by Lemma 2.6. Injectivity of the left vertical arrow implies that $u \circ B^{2s}_{L,w=s}$ is injective, so the weight-s summand meets $\ker(u)$ trivially. □

6.2. **Evaluation of θ_r on Bott elements.**

Lemma 6.2. *Let $t \geq 1$, let $q = 2^\nu$ with $\nu \geq 3$, and let L' be a field of characteristic zero containing $\sqrt{-1}$ over which $\mu_q^{\otimes t}$ is constant. For every Bott element $\mathfrak{b} \in K_{2t}(L';\mathbf{Z}/q)$ of Kahn, viewed in $\pi_{0,-t}(\mathrm{kgl}/q)(L')$ via Lemma 2.5, we have $\theta_t(\mathfrak{b}) = 0$ in $K^M_{2t}(L')/q$.*

Proof. Set $k_t = \mathbf{Q}(\mu_q)^{\ker(\chi_q^t)}$, so $k_t(\sqrt{-1}) \subseteq L'$ and $\mu_q^{\otimes t}$ is constant over $k_t(\sqrt{-1})$. Naturality of Kahn's Bott map [11, Corollary 9.5(0)] and uniqueness of lifts in Lemma 2.5 show that $\mathfrak{b}$ is restricted from $k_t(\sqrt{-1})$; since θ_t is defined over $\mathbf{Q}$, we may assume $L' = k_t(\sqrt{-1})$.

If $t \geq 2$, the totally imaginary number field L' has $\mathrm{cd}_2(L') = 2 < 2t$, so $K^M_{2t}(L')/q \cong H^{2t}_{\mathrm{et}}(L',\mu_q^{\otimes 2t}) = 0$.

Assume $t = 1$, so $L' = \mathbf{Q}(\mu_q)$ contains a primitive qth root of unity ζ. Let $P^2(q) = \mathrm{cofib}(q\colon S^1 \to S^1)$ be the mod-q Moore space, so $\Sigma^\infty P^2(q) \simeq \Sigma^{1,0}(\mathbf{1}/q)$. For every $E \in \mathrm{SH}(L')_{(2)}$, Spanier–Whitehead duality $D(\mathbf{1}/q) \simeq \Sigma^{-1,0}(\mathbf{1}/q)$ and the $(\Sigma^\infty,\Omega^\infty)$-adjunction give natural isomorphisms

$$\pi_{2,0}(E/q)(L') = [S^{2,0}, E \wedge \mathbf{1}/q] \cong [\Sigma^\infty P^2(q)_{L'}, E] \cong [P^2(q)_{L'}, \Omega^\infty E]_*.$$

Naturality along $\Sigma^{2,1}\theta_1\colon f_1\mathrm{KGL}_{(2)} \to \Sigma^{4,2}H\mathbf{Z}_{(2)}$ gives a commutative square

$$\begin{array}{ccc} \pi_{2,0}(f_1\mathrm{KGL}/q)(L') & \xrightarrow{\sim} & [P^2(q)_{L'}, BGL_{(2)}]_* \\ {\scriptstyle\theta_1}\downarrow & & \downarrow{\scriptstyle\Omega^\infty(\Sigma^{2,1}\theta_1)_*} \\ K^M_2(L')/q & \xrightarrow[\sim]{} & \widetilde{H}^{4,2}(P^2(q)_{L'},\mathbf{Z}_{(2)}). \end{array}$$

By [11, Proposition 6.3 and Corollary 9.5(ii)], every Bott element $\mathfrak{b} \in K_2(L';\mathbf{Z}/q) \cong \pi_{2,0}(f_1\mathrm{KGL}/q)(L')$ arising from Kahn's canonical Bott map corresponds to a pointed map

$$P^2(q)_{L'} \longrightarrow BGL_{1,L'} \longrightarrow BGL_{L'}.$$

By Proposition 5.3(1) and Proposition 3.7(1), we have

$$\Omega^\infty(\Sigma^{2,1}\theta_1) = \pm(c_1^2 - 2c_2),$$

which restricts to $\pm c_1^2$ on BGL_1. It therefore suffices to show that

$$x^2 = 0 \qquad \text{in } \widetilde{H}^{4,2}(P^2(q)_{L'},\mathbf{Z}_{(2)})$$

for every $x \in \widetilde{H}^{2,1}(P^2(q)_{L'},\mathbf{Z}_{(2)})$.

Choose a finite pointed 2-dimensional simplicial set K weak equivalent to $P^2(q)$. Filtering $\widetilde{H}^{a,b}(K_{L'},\mathbf{Z}_{(2)})$ by the simplicial skeleta $\mathrm{sk}_p K$ $(p = 0,1,2)$ gives a multiplicative spectral sequence

$$E_2^{p,a-p}(b) = \widetilde{H}^p_{\mathrm{sing}}(P^2(q); H^{a-p,b}(L',\mathbf{Z}_{(2)})) \Longrightarrow \widetilde{H}^{a,b}(K_{L'},\mathbf{Z}_{(2)}).$$

For any abelian group M, the group $\widetilde{H}^p_{\mathrm{sing}}(P^2(q);M)$ vanishes for $p \notin \{1,2\}$, equals $M[q]$ for $p = 1$, and equals M/qM for $p = 2$, so $E_2 = E_\infty$. Set $M_1 = H^{1,1}(L',\mathbf{Z}_{(2)}) = (L')^\times \otimes \mathbf{Z}_{(2)}$ and $M_2 = H^{2,2}(L',\mathbf{Z}_{(2)}) = K^M_2(L')_{(2)}$. Since $H^{0,1}(L',\mathbf{Z}_{(2)}) = 0$ and $H^{3,2}(L',\mathbf{Z}_{(2)}) = 0$,

only $p=1$ contributes in bidegree $(2,1)$ and only $p=2$ contributes in bidegree $(4,2)$. We therefore obtain canonical isomorphisms

$$\widetilde{H}^{2,1}(K_{L'},\mathbf{Z}_{(2)}) \xrightarrow{\sim} \widetilde{H}^1_{\mathrm{sing}}(P^2(q);M_1) \cong \mu_q(L'),$$
$$\widetilde{H}^{4,2}(K_{L'},\mathbf{Z}_{(2)}) \xrightarrow{\sim} \widetilde{H}^2_{\mathrm{sing}}(P^2(q);M_2) \cong K_2^M(L')/q,$$

under which the motivic cup square corresponds to the singular cup square induced by the Steinberg symbol pairing $\{-,-\}\colon M_1\otimes M_1\to M_2$. Every element of $\widetilde{H}^1_{\mathrm{sing}}(P^2(q);M_1)$ has the form $(\phi_\xi)_*(u_1)$, where u_1 generates $\widetilde{H}^1_{\mathrm{sing}}(P^2(q);\mathbf{Z}/q)$ and $\phi_\xi\colon \mathbf{Z}/q\to M_1$ sends 1 mod q to some $\xi=\zeta^k\in\mu_q(L')\subseteq M_1$. Its cup square in $\widetilde{H}^2_{\mathrm{sing}}(P^2(q);M_2)$ is the pushforward of $u_1\smile u_1$ along the bilinear map $\mathbf{Z}/q\otimes\mathbf{Z}/q \xrightarrow{\phi_\xi\otimes\phi_\xi} M_1\otimes M_1 \xrightarrow{\{-,-\}} M_2$, which sends $1\otimes 1$ to $\{\xi,\xi\}$. Since $-1=\zeta^{q/2}$ with $q/2$ even and $2\{-1,\zeta\}=0$, we have

$$\{\xi,\xi\}=k^2\{\zeta,\zeta\}=k^2\{-1,\zeta\}=k^2\tfrac{q}{2}\{\zeta,\zeta\}=0 \qquad \text{in } K_2^M(L').$$

Thus the coefficient map $\mathbf{Z}/q\otimes\mathbf{Z}/q\to M_2$ is zero, so $x^2=0$ and $\theta_1(\mathfrak{b})=0$. □

Proposition 6.3. *Let L be a field of characteristic zero containing $\sqrt{-1}$, let $n\geq 5$ with m,r,j as in (1.1), and let $q=2^\nu$ with $\nu\geq 3$. Identifying $\pi_{-j,-m}(\mathrm{kgl}/q)(L)$ with its image in $K_n(L;\mathbf{Z}/q)$ via Lemma 6.1, we have*

$$\theta_r(x)=2^r p_L(x) \qquad \text{in } K_n^M(L)/q \tag{6.1}$$

for every $x\in\pi_{-j,-m}(\mathrm{kgl}/q)(L)$.

Proof. By Lemma 6.1 and its proof, x is a sum of classes $\beta^{r-t}\operatorname{Tr}_{L'/L}(\mathfrak{b}\cdot\widehat{A})$ $(1\leq t\leq r)$ together with $\beta^r z$, where $z\in\pi_{-n,-n}(\mathrm{kgl}/q)(L)$ satisfies $u(z)=p_L(x)$. By Proposition 5.3(1), we have $\theta_r\circ\beta^{r-t}=2^{r-t}\,\Sigma^{2(r-t),r-t}\theta_t$, so

$$\theta_r(x)=\sum 2^{r-t}\operatorname{Tr}_{L'/L}(\theta_t(\mathfrak{b})\cdot\bar{A})+2^r u(z)=2^r p_L(x)$$

by Lemma 6.2. □

6.3. **Descent along $F(\sqrt{-1})/F$.**

Lemma 6.4. *Let F be a field of characteristic zero not containing $\sqrt{-1}$, put $L=F(\sqrt{-1})$, and let $n\geq 1$. Then*

$$\ker(\mathrm{res}_{L/F}\colon K_n^M(F)/2\longrightarrow K_n^M(L)/2)=\bar{\rho}\cdot K_{n-1}^M(F)/2.$$

Proof. Let σ generate $\mathrm{Gal}(L/F)$. In the short exact sequence of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$-modules

$$0\longrightarrow \mathbf{F}_2 \xrightarrow{1+\sigma} \mathbf{F}_2[\mathrm{Gal}(L/F)] \longrightarrow \mathbf{F}_2\longrightarrow 0,$$

Shapiro's lemma identifies $H^n_{\mathrm{et}}(F,\mathbf{F}_2)\to H^n_{\mathrm{et}}(F,\mathbf{F}_2[\mathrm{Gal}(L/F)])$ with the restriction map $H^n_{\mathrm{et}}(F,\mathbf{F}_2)\to H^n_{\mathrm{et}}(L,\mathbf{F}_2)$, and the connecting map $H^{n-1}_{\mathrm{et}}(F,\mathbf{F}_2)\to H^n_{\mathrm{et}}(F,\mathbf{F}_2)$ is cup product with the quadratic character $\chi_{L/F}$ corresponding by Kummer theory to $-1\in F^\times/F^{\times 2}$ [13, Theorem 6]. The norm residue isomorphism $K_*^M/2\xrightarrow{\sim}H^*_{\mathrm{et}}(-,\mathbf{F}_2)$ gives the claim. □

Proof of Theorem D for $n\geq 5$. Let m,r,j be as in (1.1), so $r\geq 2$, let $y\in\pi_{-j,-m}\mathrm{kgl}(F)$ be a cover class, and set $q=2^{r+1}$ $(\nu=r+1\geq 3)$. Write $\theta_r(y)\in K_n^M(F)_{(2)}$ for the image of y under θ_r. We first show that

$$\theta_r(y)\in 2^r K_n^M(F)_{(2)}+K_n^M(F)_{(2)}[2]. \tag{6.2}$$

If $\sqrt{-1}\in F$, applying Proposition 6.3 to the image of y in $\pi_{-j,-m}(\mathrm{kgl}/q)(F)$ gives

$$\theta_r(y)\in 2^r K_n^M(F)_{(2)}+qK_n^M(F)_{(2)}=2^r K_n^M(F)_{(2)}.$$

Assume $\sqrt{-1}\notin F$, and set $L=F(\sqrt{-1})$ with $\mathrm{Gal}(L/F)=\langle\sigma\rangle$. Let $a=p_L(\mathrm{res}_{L/F}(y) \bmod q)\in K_n^M(L)/q$, and choose a lift $a_0\in K_n^M(L)_{(2)}$ of a. Naturality of B_L^n under σ gives $\sigma(a)=a$, so $\sigma(a_0)\equiv a_0 \pmod{qK_n^M(L)_{(2)}}$ and hence

$$\mathrm{res}_{L/F}(N_{L/F}(a_0))=a_0+\sigma(a_0)\equiv 2a_0\equiv 0 \pmod{2K_n^M(L)_{(2)}}.$$

By Lemma 6.4, $N_{L/F}(a_0) = \rho c + 2c'$ for some $c \in K^M_{n-1}(F)_{(2)}$ and $c' \in K^M_n(F)_{(2)}$. Meanwhile, Proposition 6.3 gives $\mathrm{res}_{L/F}(\theta_r(y)) = 2^r a_0 + qz$ for some $z \in K^M_n(L)_{(2)}$. Since $N_{L/F} \circ \mathrm{res}_{L/F} = 2$ and $2\rho = 0$, applying $N_{L/F}$ gives

$$2\theta_r(y) = 2^r(\rho c + 2c') + qN_{L/F}(z) = 2^{r+1}\big(c' + N_{L/F}(z)\big),$$

so $\theta_r(y) - 2^r(c' + N_{L/F}(z))$ lies in $K^M_n(F)_{(2)}[2]$. This proves (6.2).

By Proposition 5.8, the image of $c^{\mathrm{st}}_n(y)$ in $K^M_n(F)_{(2)}$ equals $\pm\frac{(n-1)!}{2^r}\theta_r(y)$, and $\frac{(n-1)!}{2^r}$ is an even integer by Proposition 5.3(2). Multiplying (6.2) by $\frac{(n-1)!}{2^r}$ annihilates $K^M_n(F)_{(2)}[2]$ and places the image of $c^{\mathrm{st}}_n(y)$ in $(n-1)!\,K^M_n(F)_{(2)} = 2^{v_2((n-1)!)}\,K^M_n(F)_{(2)}$. □

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