LOCALIZING MOTIVES OVER NONZERO CONNECTIVE $\mathbb{E}_2$-RINGS ARE NOT COMPACTLY GENERATED

Abstract. We prove that Efimov's category of finitary localizing motives over any nonzero connective $\mathbb{E}_2$-ring spectrum is not compactly generated; more generally, this holds over every $\mathbb{E}_2$-ring that admits an $\mathbb{E}_2$-map to a field. Specifically, the motive of the free algebra on one generator in degree -2 does not lie in the localizing subcategory generated by the compact motives, removing the rationality hypothesis in a recent theorem of Ramzi. Over a perfect field of positive characteristic p, we construct an explicit nonzero motive that is right orthogonal to all compact motives. It is built from the fiber of the mod-p cyclotomic trace on the square-zero extension $k \oplus \Sigma k$: orthogonality to compact motives follows from a Dundas–Goodwillie–McCarthy theorem with compact motivic coefficients, while nontriviality is detected on the test algebra via Artin–Schreier classes in topological cyclic homology.

1. Introduction

1.1. **Results.** For an $\mathbb{E}_2$-ring spectrum R, the category Mod_R of right R-modules is $\mathbb{E}_1$-monoidal [12, Corollary 5.1.2.6]. Let $\mathrm{Mot}_R^{\mathrm{loc}}$ denote Efimov's category of finitary localizing motives over Mod_R [7, Sections 1.5–1.6], the target of the universal finitary localizing invariant

$$U_R \colon \{\text{small idempotent-complete stable categories with a left } \mathrm{Perf}(R)\text{-action}\} \longrightarrow \mathrm{Mot}_R^{\mathrm{loc}}\,.$$

For an algebra A in Mod_R, its category of perfect right modules $\mathrm{Perf}(A)$ inherits a left $\mathrm{Perf}(R)$-action, and we abbreviate $U_R(A) = U_R(\mathrm{Perf}(A))$. For an ordinary commutative ring B, we write HB for its Eilenberg–Mac Lane spectrum, and often abbreviate $\mathrm{Mot}_B^{\mathrm{loc}} = \mathrm{Mot}_{HB}^{\mathrm{loc}}$ and $U_B = U_{HB}$.

Efimov proved that $\mathrm{Mot}_R^{\mathrm{loc}}$ is dualizable as a presentable stable category, and that it is rigid whenever Mod_R is $\mathbb{E}_2$-monoidal, for instance when R is an $\mathbb{E}_3$-ring [7, Theorems 0.3 and 3.1]; in the rigid case the compact objects of $\mathrm{Mot}_R^{\mathrm{loc}}$ coincide with its dualizable objects. Dualizability and rigidity, however, do not imply compact generation. Efimov showed that $\mathrm{Mot}_{\mathbb{Q}[x]}^{\mathrm{loc}}$ is not compactly generated [7, Corollary 12.11], and Ramzi established the same for $\mathrm{Mot}_R^{\mathrm{loc}}$ whenever R is a connective $\mathbb{E}_2$-ring with $R \otimes \mathbb{Q} \neq 0$, in particular for the sphere spectrum $R = \mathbb{S}$ [16]. Ramzi's argument relies on Mathew's degeneration theorem, which forces the circle action on the Hochschild homology of a dualizable motive over a characteristic-zero field to be trivial; as Ramzi observes, this method does not directly apply to ring spectra supported in positive characteristic, such as $\mathbb{F}_p$-algebras. Our main result removes the rationality assumption completely.

For a presentable stable category $\mathcal{D}$, write $\mathrm{Loc}(\mathcal{D}^\omega)$ for the localizing subcategory generated by its compact objects, i.e., the smallest full stable subcategory containing $\mathcal{D}^\omega$ and closed under colimits. Let $\mathbb{S}[t] = T_{\mathbb{S}}(\Sigma^{-2}\mathbb{S})$ be the free $\mathbb{E}_1$-ring on a generator t in homological degree -2, and set $R[t] = R \otimes \mathbb{S}[t]$, the base-changed algebra in Mod_R. For a field κ, the κ-algebra $(H\kappa)[t]$ is the graded polynomial algebra $\kappa[t]$ on a generator of degree -2 with zero differential (Lemma 2.1).

Theorem A. *Let R be an $\mathbb{E}_2$-ring that admits an $\mathbb{E}_2$-map $R \to H\kappa$ to a field κ; in particular, this holds for every nonzero connective $\mathbb{E}_2$-ring. Then*

$$U_R(R[t]) \notin \mathrm{Loc}((\mathrm{Mot}_R^{\mathrm{loc}})^{\omega}).$$

Consequently, $\mathrm{Mot}_R^{\mathrm{loc}}$ *contains a nonzero object that is right orthogonal to every compact object, and* $\mathrm{Mot}_R^{\mathrm{loc}}$ *is not compactly generated.*

Base change along $R \to H\kappa$ preserves compact motives and colimits, reducing Theorem A to the case of a field.

Theorem B. *For every field κ, the motive $U_\kappa(\kappa[t])$, with $|t| = -2$, does not lie in* $\mathrm{Loc}((\mathrm{Mot}_\kappa^{\mathrm{loc}})^{\omega})$.

In characteristic zero, Theorem B follows immediately from Ramzi's criterion, since the weight-one summand of $\mathrm{HH}(\kappa[t]/\kappa)$ is a shift of the free circle module $\kappa[\mathbb{T}]$. In positive characteristic, base change to the perfect closure reduces the problem to a perfect field k of characteristic $p > 0$, where we construct an explicit nonzero object in the right orthogonal to $(\mathrm{Mot}_k^{\mathrm{loc}})^{\omega}$.

Recall that $\mathrm{Mot}_k^{\mathrm{loc}}$ is rigid and self-dual: for a compact (hence dualizable) motive $P \in \mathrm{Mot}_k^{\mathrm{loc}}$ with dual $P^\vee$, corepresentability of nonconnective K-theory gives $\mathrm{Map}_{\mathrm{Mot}_k^{\mathrm{loc}}}(P, Z) \simeq K(P^\vee \otimes Z)$. Moreover, Efimov's self-duality equivalence $\mathrm{Mot}_k^{\mathrm{loc}} \simeq \mathrm{Fun}^L(\mathrm{Mot}_k^{\mathrm{loc}}, \mathrm{Sp})$, $X \mapsto K(X \otimes -)$ [7, Theorem 3.1(i)], represents every colimit-preserving spectrum-valued functor on $\mathrm{Mot}_k^{\mathrm{loc}}$ by an object of $\mathrm{Mot}_k^{\mathrm{loc}}$. We show that $X \mapsto \mathrm{TC}(\mathrm{THH}(X))/p$ preserves colimits on $\mathrm{Mot}_k^{\mathrm{loc}}$, so there is an object $J \in \mathrm{Mot}_k^{\mathrm{loc}}$ representing the fiber of the mod-p cyclotomic trace:

$$K(J \otimes X) \simeq \mathrm{fib}\big(K(X)/p \longrightarrow \mathrm{TC}(\mathrm{THH}(X))/p\big) \qquad (X \in \mathrm{Mot}_k^{\mathrm{loc}}).$$

Let $\widetilde{U}(k \oplus \Sigma k) = \mathrm{fib}(U(k \oplus \Sigma k) \to U(k))$ be the reduced motive of the trivial square-zero extension $k \oplus \Sigma k$, and set

$$Z = J \otimes \widetilde{U}(k \oplus \Sigma k).$$

Theorem C. *Let k be a perfect field of characteristic $p > 0$. For every compact $P \in \mathrm{Mot}_k^{\mathrm{loc}}$, we have $K(Z \otimes P) \simeq 0$ and $\mathrm{Map}_{\mathrm{Mot}_k^{\mathrm{loc}}}(P, Z) \simeq 0$, whereas*

$$\pi_{-3} K\big(Z \otimes U(k[t])\big) \neq 0.$$

In particular $Z \neq 0$, and $\mathrm{Mot}_k^{\mathrm{loc}}$ *is not compactly generated.*

Theorem A also applies to nonconnective $\mathbb{E}_2$-rings that admit a field point, such as augmented $\mathbb{E}_2$-algebras over a field. By contrast, if an $\mathbb{E}_2$-ring R has a unit in a nonzero homotopy degree, it admits no ring map to $H\kappa$ for any field κ (as the unit and its inverse would map to zero while their product maps to 1), leaving periodic ring spectra outside the scope of Theorem A.

1.2. **Main ideas of the proof.** The reduction of Theorem A to Theorem B is formal: an $\mathbb{E}_2$-map $R \to S$ induces an $\mathbb{E}_1$-monoidal base change on module categories and hence a functor $f_!\colon \mathrm{Mot}_R^{\mathrm{loc}} \to \mathrm{Mot}_S^{\mathrm{loc}}$ whose right adjoint preserves colimits. Thus $f_!$ preserves compact objects and maps $\mathrm{Loc}((\mathrm{Mot}_R^{\mathrm{loc}})^{\omega})$ into $\mathrm{Loc}((\mathrm{Mot}_S^{\mathrm{loc}})^{\omega})$, while sending $U_R(R[t])$ to $U_S(S[t])$. When R is nonzero and connective, $\pi_0 R$ is a nonzero commutative ring, and quotienting by a maximal ideal yields an $\mathbb{E}_2$-map $R \to H\pi_0 R \to H\kappa$.

The heart of the paper is Theorem C, whose proof has two parts.

Part 1: Orthogonality to compact motives. For a compact motive $P \in \mathrm{Mot}_k^{\mathrm{loc}}$, the spectrum $K(Z \otimes P)$ is the relative term of the localizing invariant $E(-) = K(J \otimes P \otimes U(-))$ along

the split augmentation $k \oplus \Sigma k \to k$. Showing that it vanishes amounts to proving a Dundas–Goodwillie–McCarthy theorem with coefficients in P (Proposition 5.6). To avoid functor-calculus hypotheses that are not automatic after twisting by an abstract motive P, we prove a self-contained truncation criterion (Theorem 3.1): a localizing invariant E of k-linear categories vanishes on all maps $A \to B$ of connective k-algebras with 1-connective fiber (and is therefore truncating) provided it satisfies two properties:

(1) *Uniform continuity.* The connectivity of $E(A,B) = \mathrm{fib}(E(A) \to E(B))$ tends to $+\infty$ with the connectivity of $\mathrm{fib}(A \to B)$, uniformly over all maps of connective k-algebras. For $K(P \otimes U(-))$, we deduce this from Efimov's theorem that U preserves infinite products [7, Corollary 4.2], combined with a k-linear form of Córdova Fedeli's comparison between products of additive categories and products of their stabilizations (Proposition 4.2): a sequence of maps with arbitrarily highly connected fibers whose relative classes stayed below a fixed degree could be shifted into a single degree by tensoring with Laurent polynomial rings, producing an element in an infinite product not coming from the direct sum (Proposition 4.1).

(2) *Vanishing derivative.* For every connective k-algebra A and connective A-bimodule M, the stabilization $\mathrm{colim}_n\, \Omega^n E(A \oplus \Sigma^n M, A)$ vanishes. Using the identification of the Goodwillie derivative of laced K-theory with THH due to Harpaz, Nikolaus, and Saunier [9], together with Efimov's generation of $\mathrm{Mot}_k^{\mathrm{loc}}$ by proper categories (Lemma 5.3), we identify the stabilization of $K(X \otimes U(A \oplus \Sigma^n M), X \otimes U(A))$ for any $X \in \mathrm{Mot}_k^{\mathrm{loc}}$ with $\Sigma(\mathrm{THH}(X) \otimes_{\mathrm{THH}(k)} \mathrm{THH}(A;M))$ (Lemma 5.2). Taking $A = M = k$ and invoking the classical Dundas–Goodwillie–McCarthy theorem shows that $\mathrm{THH}(J) \simeq 0$ (Lemma 5.5). Symmetric monoidality of THH gives $\mathrm{THH}(J \otimes P) \simeq 0$, killing the derivative of $K(J \otimes P \otimes U(-))$.

The truncation criterion itself proceeds by an induction on connectivity using two fiber sequences of Land and Tamme [10, 11]. Given $A \to B$ with d-connective fiber, the $\odot$-construction and the tensor-algebra pullback square reduce $E(A,B)$ to split square-zero extensions with $(d-1)$-connective ideal and auxiliary maps of connectivity at least $2d$, while uniform continuity terminates the iteration. A first pass yields an eventual linear bound $d - D_0$ (Lemma 3.2), and the vanishing of the stabilization repeatedly doubles the slope of this linear bound while halving the connectivity threshold down to 1 (Lemma 3.3).

Part 2: Detection on the test algebra. To prove $\pi_{-3}K(Z \otimes U(k[t])) \neq 0$, we examine the fiber sequence

$$K(Z \otimes U(k[t])) \longrightarrow K(k[t] \oplus \Sigma k[t], k[t])/p \longrightarrow \mathrm{TC}(k[t] \oplus \Sigma k[t], k[t])/p.$$

Notice that $k[t]$ is not connective, so Proposition 5.6 does not apply. On the K-theory side, we combine localization sequences with the coconnective polynomial dévissage theorem of Burklund and Levy [4] and the Bass exact sequence to prove that $K(k[t] \oplus \Sigma k[t], k[t])$ is connective (Proposition 6.2). On the TC side, Land and Tamme's tensor-algebra formula trades the relative TC/p of $k[t] \oplus \Sigma k[t] \to k[t]$ (up to a desuspension) for that of $k[t] \otimes_k k[x] \to k[t]$, where $|x| = 2$. In the two-variable cyclic bar construction, words containing n letters t and n letters x lie in total degree $-2n + 2n = 0$: their representation spheres $S^{-2\rho_n}$ and $S^{2\rho_n}$ cancel compatibly with the cyclotomic Frobenius, splitting off a cyclotomic summand equivalent to the reduced THH of the discrete polynomial ring $k[y]$ ($|y| = 0$) tensored with $(\Sigma_+^\infty \mathbb{T})^{\mathrm{triv}}$ (Lemma 6.4). Finally, the class of y in the Artin–Schreier cokernel $yk[y]/\{f^p - f : f \in yk[y]\}$ produces a nonzero class in $\pi_{-1}\widetilde{\mathrm{TC}}(k[y])/p$, hence in $\pi_{-2}\,\mathrm{TC}(k[t] \oplus \Sigma k[t], k[t])/p$ (Proposition 6.3), which crosses the boundary map to give a nonzero element of $\pi_{-3}K(Z \otimes U(k[t]))$.

The degree $|t| = -2$ is tailored to both halves of this calculation: evenness ensures commutativity and the vanishing of class differences in K_0 needed for connectivity of relative K-theory, while negativity cancels the degree-2 tensor-algebra generator x along the diagonal so that Artin–Schreier theory creates a class in negative degree below the K-theory range.

1.3. **Conventions.** Categories are ∞-categories. Compactness means ω-compactness. For a spectrum X, we write $X \geq d$ if $\pi_i X = 0$ for all $i < d$. In any stable category, X/p denotes the cofiber of multiplication by p on X. We write K for nonconnective algebraic K-theory. For a map of algebras $A \to B$ and a localizing invariant E, we set $E(A) = E(\mathrm{Perf}(A))$ and $E(A, B) = \mathrm{fib}(E(A) \to E(B))$.

In Sections 3–6, k denotes a fixed perfect field of characteristic $p > 0$, also viewed as its Eilenberg–Mac Lane ring spectrum Hk; algebras are $\mathbb{E}_1$-algebras in Mod_k, bimodules over A are $A \otimes_k A^{\mathrm{op}}$-modules, and we write $U = U_k$ and $\mathbf{1} = U(k)$.

We write CycSp for the category of Nikolaus–Scholze cyclotomic spectra with Frobenius maps at all primes [13]; any restriction to p-typical cyclotomic spectra is stated explicitly. Theorem and section references to [13] follow arXiv v2, incorporating the published corrections in [14] (in particular the pointwise tensor product in the functor-category construction of the Tate diagonal [14, correction to p. 301]).

1.4. **Outline.** Section 2 reduces Theorem A to Theorem B and establishes the characteristic-zero case of Theorem B. Section 3 proves the abstract truncation criterion (Theorem 3.1). Section 4 establishes uniform continuity for K-theory with compact motivic coefficients (Proposition 4.1). Section 5 computes the derivative of K-theory with coefficients, constructs the trace-fiber motive J, and proves the Dundas–Goodwillie–McCarthy theorem with compact coefficients (Proposition 5.6). Section 6 carries out the relative K-theory and $\mathrm{TC}\,/p$ computations on $k[t]$ and completes the proofs of Theorems C and B. Appendix A records the commutation of homotopy orbits and Tate constructions with uniformly bounded-below limits used for TR.

2. Base change of localizing motives

In this section we reduce Theorem A to the field case (Theorem B) and prove Theorem B in characteristic zero, following Efimov [7] and Ramzi [16].

Recall that for any presentable stable category $\mathcal{D}$, the inclusion $\mathcal{D}^\omega \hookrightarrow \mathcal{D}$ of the compact objects extends to a colimit-preserving fully faithful functor $\mathrm{Ind}(\mathcal{D}^\omega) \to \mathcal{D}$ whose essential image is $\mathrm{Loc}(\mathcal{D}^\omega)$.

2.1. **Base change.** Let $f\colon R \to S$ be a map of $\mathbb{E}_2$-rings. Extension of scalars defines a colimit-preserving $\mathbb{E}_1$-monoidal functor $\mathrm{Mod}_R \to \mathrm{Mod}_S$ [12, Corollary 5.1.2.6]. Note that Mod_R is rigid, as its monoidal unit R is compact and generates Mod_R under shifts and colimits [7, Proposition 1.1]. Base change of small idempotent-complete stable categories with a left $\mathrm{Perf}(R)$-action along $\mathrm{Perf}(R) \to \mathrm{Perf}(S)$ descends to a colimit-preserving functor

$$f_!\colon \mathrm{Mot}_R^{\mathrm{loc}} \longrightarrow \mathrm{Mot}_S^{\mathrm{loc}},$$

which is strongly continuous by [7, Proposition 5.1]: its right adjoint $f^*\colon \mathrm{Mot}_S^{\mathrm{loc}} \to \mathrm{Mot}_R^{\mathrm{loc}}$, which sends $U_S(\mathcal{C})$ to $U_R(\mathcal{C}|_{\mathrm{Perf}(R)})$ by restricting the module action, also preserves colimits. Consequently $f_!$ preserves compact objects, and since $f_!$ preserves colimits we obtain

$$f_!(\mathrm{Loc}((\mathrm{Mot}_R^{\mathrm{loc}})^\omega)) \subseteq \mathrm{Loc}((\mathrm{Mot}_S^{\mathrm{loc}})^\omega). \tag{2.1}$$

By the tensor product formula for module categories [12, Theorem 4.8.5.16 and Remark 4.8.5.17], there is an equivalence $\mathrm{Mod}_R \otimes_{\mathrm{Sp}} \mathrm{Mod}_{\mathbb{S}[t]} \simeq \mathrm{Mod}_{R[t]}$ of left Mod_R-modules, identifying $U_R(R[t])$

with the base change from $\mathbb{S}$ to R of $U_{\mathbb{S}}(\mathbb{S}[t])$. By associativity of base change, $f_! U_R(R[t]) \simeq U_S(S[t])$.

Lemma 2.1. *For a field κ, the algebra $(H\kappa)[t] = H\kappa \otimes \mathbb{S}[t]$ is equivalent to the graded polynomial algebra $\kappa[t]$ on a generator of degree -2 with zero differential.*

Proof. As an $\mathbb{E}_1$-ring over $\mathbb{S}$, the tensor algebra $\mathbb{S}[t]$ has underlying spectrum $\bigoplus_{n\geq 0}(\Sigma^{-2}\mathbb{S})^{\otimes n}$. Extension of scalars along $\mathbb{S} \to H\kappa$ is symmetric monoidal and sends it to the tensor algebra in Mod_κ on $\Sigma^{-2}H\kappa$, which is the graded polynomial algebra $\kappa[t]$ with $|t| = -2$ and zero differential. □

Proof of Theorem A, assuming Theorem B. Let $f\colon R \to H\kappa$ be an $\mathbb{E}_2$-map to a field κ. If $U_R(R[t])$ lay in $\mathrm{Loc}((\mathrm{Mot}_R^{\mathrm{loc}})^\omega)$, then by (2.1) and Lemma 2.1 its base change $f_! U_R(R[t]) \simeq U_\kappa(\kappa[t])$ would lie in $\mathrm{Loc}((\mathrm{Mot}_\kappa^{\mathrm{loc}})^\omega)$, contradicting Theorem B.

Consider the Verdier quotient $\mathrm{Mot}_R^{\mathrm{loc}} \to \mathrm{Mot}_R^{\mathrm{loc}} / \mathrm{Loc}((\mathrm{Mot}_R^{\mathrm{loc}})^\omega)$ in presentable stable categories. It is a localization whose right adjoint is fully faithful with essential image the full subcategory of objects right orthogonal to $\mathrm{Loc}((\mathrm{Mot}_R^{\mathrm{loc}})^\omega)$, equivalently to $(\mathrm{Mot}_R^{\mathrm{loc}})^\omega$ [3, Propositions 5.6 and 5.9]. Since $U_R(R[t]) \notin \mathrm{Loc}((\mathrm{Mot}_R^{\mathrm{loc}})^\omega)$, its image in the Verdier quotient is nonzero; applying the right adjoint produces a nonzero object of $\mathrm{Mot}_R^{\mathrm{loc}}$ that is right orthogonal to every compact object. In particular, $\mathrm{Mot}_R^{\mathrm{loc}}$ is not compactly generated.

Finally, suppose R is a nonzero connective $\mathbb{E}_2$-ring. Since the unit of R acts as the identity on $\pi_* R$, the commutative ring $\pi_0 R$ is nonzero. Postnikov truncation gives an $\mathbb{E}_2$-map $R \to H\pi_0 R$ [12, Proposition 7.1.3.15]. Choosing a maximal ideal $\mathfrak{m} \subset \pi_0 R$ with residue field $\kappa = \pi_0 R/\mathfrak{m}$, the equivalence between discrete $\mathbb{E}_2$-rings and ordinary commutative rings [12, Proposition 7.1.3.18] provides an $\mathbb{E}_2$-map $H\pi_0 R \to H\kappa$, and the composite $R \to H\pi_0 R \to H\kappa$ is a field point. □

2.2. **Characteristic zero.**

Theorem 2.2 (Ramzi [16, Corollary 2.3])**.** *Let κ be a field of characteristic zero and let $N \in \mathrm{Mot}_\kappa^{\mathrm{loc}}$. If $\mathrm{HH}(N/\kappa)$, equipped with its circle action, admits the free circle module $\kappa[\mathbb{T}] = \kappa \otimes \Sigma^\infty_+ \mathbb{T}$ as a retract in $\mathrm{Mod}_\kappa^{B\mathbb{T}}$, then $N \notin \mathrm{Loc}((\mathrm{Mot}_\kappa^{\mathrm{loc}})^\omega)$.*

Proof of Theorem B in characteristic zero. Relative Hochschild homology $\mathrm{HH}(\kappa[t]/\kappa)$ splits in $\mathrm{Mod}_\kappa^{B\mathbb{T}}$ according to the word length in t. In the cyclic bar construction, the weight-one summand has simplicial degree q given by $q+1$ copies of $\Sigma^{-2}\kappa$, one for each slot occupied by t: face maps merge adjacent slots, degeneracies insert units, and the cyclic operator rotates the slots. Thus the indexing cyclic set is the representable cyclic set $\Lambda([0])$ (a cyclic map to $[0]$ is determined by the unique jump of a monotone $\mathbb{Z}$-equivariant lift in one period), whose geometric realization is $\mathbb{T}$ with the translation circle action [13, Construction B.9 and Remark B.11]. Consequently,

$$\mathrm{HH}(\kappa[t]/\kappa)_{(1)} \simeq \Sigma^{-2}\kappa[\mathbb{T}]$$

in $\mathrm{Mod}_\kappa^{B\mathbb{T}}$. Since $\mathrm{HH}(-/\kappa)$ is exact, $\mathrm{HH}(\Sigma^2 U_\kappa(\kappa[t])/\kappa)$ admits $\kappa[\mathbb{T}]$ as a retract, so $\Sigma^2 U_\kappa(\kappa[t])$—and hence $U_\kappa(\kappa[t])$—does not lie in $\mathrm{Loc}((\mathrm{Mot}_\kappa^{\mathrm{loc}})^\omega)$ by Theorem 2.2. □

The positive-characteristic case of Theorem B is completed at the end of Section 6.

3. A truncation criterion

The Dundas–Goodwillie–McCarthy theorem asserts that the fiber of the cyclotomic trace is invariant under maps of connective rings that induce isomorphisms on π_0. In this section we prove a general criterion for a localizing invariant to be truncating in this sense, requiring only

a uniform connectivity bound and the vanishing of the first Goodwillie-type stabilization along square-zero extensions. We apply this criterion in Section 5 to trace fibers twisted by compact motives.

Throughout this section, E denotes a spectrum-valued localizing invariant of small k-linear categories. For a connective k-algebra A and an A-bimodule M, write $A \oplus M$ for the trivial square-zero extension. Because the augmentation $A \oplus M \to A$ is split by the unit $A \to A \oplus M$, the relative term $E(A \oplus M, A)$ is a summand of $E(A \oplus M)$. Applying $A \oplus -$ to the suspension pullback square $M \simeq 0 \times_{\Sigma M} 0$ of A-bimodules yields a commutative square of k-algebras

$$\begin{array}{ccc} A \oplus M & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \oplus \Sigma M, \end{array} \tag{3.1}$$

in which both maps $A \oplus M \to A$ are the augmentation and both maps $A \to A \oplus \Sigma M$ are the zero section. Taking horizontal fibers after applying E induces the suspension assembly map $E(A \oplus M, A) \to E(A, A \oplus \Sigma M) \simeq \Omega E(A \oplus \Sigma M, A)$, and hence a sequential system $\{\Omega^n E(A \oplus \Sigma^n M, A)\}_{n \geq 0}$ whose filtered colimit we call the *stabilization* of E at (A, M).

Theorem 3.1. *Suppose that E satisfies the following two conditions:*

(1) (Uniform continuity.) *For every $N \in \mathbb{Z}$ there exists an integer d_N such that for every map $A \to B$ of connective k-algebras,*

$$\operatorname{fib}(A \to B) \geq d_N \quad \Longrightarrow \quad E(A, B) \geq N. \tag{3.2}$$

(2) (Vanishing derivative.) *For every connective k-algebra A and every connective A-bimodule M,*

$$\operatorname*{colim}_{n \geq 0} \Omega^n E(A \oplus \Sigma^n M, A) \simeq 0. \tag{3.3}$$

Then $E(A, B) \simeq 0$ for every map $A \to B$ of connective k-algebras with $\operatorname{fib}(A \to B) \geq 1$. In particular, E is truncating: $E(A) \to E(\pi_0 A)$ is an equivalence for every connective k-algebra A.

The proof is an induction on the connectivity of $\operatorname{fib}(A \to B)$ driven by two fiber sequences from the work of Land and Tamme [10, 11]. Together, they bound $E(A, B)$ in terms of split square-zero extensions of connectivity one lower and maps of roughly twice the connectivity. Iterating them produces a linear connectivity bound (Lemma 3.2), after which the vanishing derivative repeatedly doubles the slope of the bound (Lemma 3.3).

3.1. **Two fiber sequences.** First, let $f\colon A \to B$ be a map of connective k-algebras with $I = \operatorname{fib}(A \to B) \geq d \geq 0$, and form the pullback $C = A \times_B A$. Both projections $p_1, p_2\colon C \to A$ are split by the diagonal $\Delta\colon A \to C$, and their fibers are equivalent to I. By the pullback theorem of Land and Tamme [10, Theorem 1.3 and Remark 1.18], there is a connective k-algebra $A \odot_C^B A$ factoring f on both copies of A such that every localizing invariant sends the square

$$\begin{array}{ccc} C & \xrightarrow{p_1} & A \\ {\scriptstyle p_2}\downarrow & & \downarrow \\ A & \longrightarrow & A \odot_C^B A \end{array}$$

to a pullback square. Consequently $E(C,A) \simeq E(A, A \odot_C^B A)$ (via the first projection $p_1 \colon C \to A$), and the fiber sequence associated to the composite $A \to A \odot_C^B A \to B$ takes the form

$$E(C,A) \longrightarrow E(A,B) \longrightarrow E(A \odot_C^B A, B). \tag{3.4}$$

We claim that the map $A \odot_C^B A \to B$ has substantially higher connectivity:

$$\mathrm{fib}(A \odot_C^B A \to B) \geq 2d+2. \tag{3.5}$$

To verify (3.5), write A_1 and A_2 for A viewed as a C-module via p_1 and p_2, respectively, and set $I_1 = \mathrm{fib}(p_2)$ and $I_2 = \mathrm{fib}(p_1)$, with their induced C-module structures. On underlying k-modules, $A \odot_C^B A$ is $A_1 \otimes_C A_2$, and $A \odot_C^B A \to B$ is induced by f and multiplication on B [10, Theorem 1.3]. Consider the multiplication map $\mu \colon A_1 \otimes_C A_1 \to A_1$. On underlying modules, the map $A_1 \simeq A_1 \otimes_C C \xrightarrow{\mathrm{id} \otimes p_1} A_1 \otimes_C A_1$ is a section of μ whose cofiber is $A_1 \otimes_C \Sigma\, \mathrm{fib}(p_1) \simeq A_1 \otimes_C \Sigma I_2$. Connectivity of A_1 and C, together with $I \geq d$, gives $\mathrm{fib}(\mu) \simeq \mathrm{cofib}(\mathrm{id} \otimes p_1) \geq d+1$. On the other hand, in the cofiber sequence $I_1 \to C \xrightarrow{p_2} A_2$ of left C-modules, the left C-action on I_1 factors through $p_1 \colon C \to A_1$. Tensoring on the left with $A_1 \otimes_C -$ and mapping to $I \to A_1 \xrightarrow{f} B$ yields a map of cofiber sequences

$$\begin{array}{ccccc} (A_1 \otimes_C A_1) \otimes_{A_1} I & \longrightarrow & A_1 & \longrightarrow & A \odot_C^B A \\ \downarrow{\scriptstyle \mu \otimes \mathrm{id}_I} & & \| & & \downarrow \\ I & \longrightarrow & A_1 & \xrightarrow[f]{} & B. \end{array}$$

The fiber of the left vertical map is $\mathrm{fib}(\mu) \otimes_{A_1} I \geq (d+1) + d = 2d+1$, so taking vertical fibers identifies $\mathrm{fib}(A \odot_C^B A \to B)$ with $\Sigma(\mathrm{fib}(\mu) \otimes_{A_1} I) \geq 2d+2$, proving (3.5).

Second, let $B \to A$ be a split augmentation of connective k-algebras with fiber $I \geq d \geq 1$. Viewing I as an A-bimodule via the section $A \to B$, a theorem of Land and Tamme [11, Theorem 4.1] states that the canonical square of k-algebras with vertices $A \oplus \Omega I$, A, A, and the tensor algebra $T_A(I) = \bigoplus_{q \geq 0} I^{\otimes_A q}$ is a motivic pullback square, giving

$$E(T_A(I), A) \simeq \Sigma E(A \oplus \Omega I, A). \tag{3.6}$$

On underlying A-bimodules, the splitting gives $B \simeq A \oplus I$. Setting $W = \bigoplus_{q \geq 2} I^{\otimes_A q}$ and letting $m \colon W \to B$ be the sum of the iterated multiplication maps, the induced k-algebra map $T_A(I) \to B$ over and under A is identified on underlying bimodules with

$$T_A(I) \simeq B \oplus W \xrightarrow{(\mathrm{id}_B, m)} B.$$

Its fiber is identified with W via $(-m, \mathrm{id}_W) \colon W \to B \oplus W$, and since A is connective and $I \geq d$, we have $W \geq 2d$. Applying E to the composite $T_A(I) \to B \to A$ and substituting (3.6) gives a fiber sequence

$$E(T_A(I), B) \longrightarrow \Sigma E(A \oplus \Omega I, A) \longrightarrow E(B, A), \tag{3.7}$$

whose second map is surjective on π_n whenever $E(T_A(I), B) \geq n$.

3.2. **A linear bound.** By uniform continuity (3.2), we may fix an integer $D_0 \geq 1$ such that $E(A,B) \geq 0$ whenever $\mathrm{fib}(A \to B) \geq D_0$. For $d \geq D_0$, let $\nu(d) \in \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ be the largest n such that $E(A,B) \geq n$ for every map $A \to B$ of connective k-algebras with $\mathrm{fib}(A \to B) \geq d$. Then ν is nondecreasing and $\nu(d) \to +\infty$ as $d \to +\infty$.

Lemma 3.2. *For every $d > D_0$, we have $\nu(d) \geq \nu(d-1) + 1$. Consequently,*

$$\mathrm{fib}(A \to B) \geq d \geq D_0 \quad \Longrightarrow \quad E(A,B) \geq d - D_0. \tag{3.8}$$

Proof. We may assume that $n = \nu(d)$ is finite. By definition of $\nu(d)$, there exist a map $A \to B$ of connective k-algebras with $\mathrm{fib}(A \to B) \geq d$ and a nonzero class $x \in \pi_n E(A, B)$. If the image of x in $\pi_n E(A \odot_C^B A, B)$ under (3.4) is nonzero, we replace $A \to B$ by $A \odot_C^B A \to B$, whose fiber is $(2d+2)$-connective by (3.5), and repeat. Since the connectivity of the fiber strictly increases at each step, uniform continuity (3.2) ensures that after finitely many replacements the right-hand term in (3.4) is $(n+1)$-connective, so the image of the class vanishes.

At that terminal step we obtain a map $A' \to B'$ with fiber $I \geq d' \geq d$ and a nonzero class in $\pi_n E(A', B')$ lifting to a nonzero class in $\pi_n E(C', A')$, where $C' = A' \times_{B'} A' \to A'$ is a split augmentation with fiber $I \geq d' \geq 2$. In (3.7) applied to $C' \to A'$, the map $T_{A'}(I) \to C'$ has $2d'$-connective fiber, so $E(T_{A'}(I), C') \geq \nu(2d') \geq \nu(d) = n$. The class in $\pi_n E(C', A')$ therefore lifts to a nonzero element of $\pi_n \Sigma E(A' \oplus \Omega I, A') \cong \pi_{n-1} E(A' \oplus \Omega I, A')$. Since $\mathrm{fib}(A' \oplus \Omega I \to A') = \Omega I \geq d' - 1 \geq d - 1 \geq D_0$, this forces $\nu(d-1) \leq n-1$, proving $\nu(d) \geq \nu(d-1) + 1$. Induction from $\nu(D_0) \geq 0$ yields (3.8). □

3.3. **Doubling the slope.** We now incorporate the vanishing derivative (3.3). Let M be a connective A-bimodule with $M \geq d \geq 0$, and apply (3.4) to the zero section $A \to A \oplus \Sigma M$, whose fiber is M. Because the trivial square-zero extension functor $A \oplus -$ is a right adjoint [11, Section 4.1], it preserves pullbacks and identifies $A \times_{A \oplus \Sigma M} A$ with $A \oplus M$ via (3.1). Factoring (3.1) through $A \odot_{A \oplus M}^{A \oplus \Sigma M} A$ identifies the first map in (3.4) with the suspension assembly map on horizontal fibers, giving a fiber sequence

$$E(A \oplus M, A) \longrightarrow \Omega E(A \oplus \Sigma M, A) \longrightarrow E\big(A \odot_{A \oplus M}^{A \oplus \Sigma M} A,\ A \oplus \Sigma M\big), \tag{3.9}$$

in which the map of algebras in the third term has $(2d+2)$-connective fiber by (3.5).

Lemma 3.3. *Let $a \geq 1$, $c \in \mathbb{Z}$, and $D \geq 1$ be integers, and suppose that $E(A, B) \geq ad - c$ whenever $\mathrm{fib}(A \to B) \geq d \geq D$. Then $E(A, B) \geq 2ad - c$ whenever $\mathrm{fib}(A \to B) \geq d \geq \max\{1, \lceil D/2 \rceil\}$.*

Proof. First let $M \geq d \geq 0$ be a connective A-bimodule with $2d+2 \geq D$, and put $X_n = \Omega^n E(A \oplus \Sigma^n M, A)$. Applying (3.9) to $\Sigma^n M \geq d + n$ and looping n times shows that $\mathrm{fib}(X_n \to X_{n+1})$ is at least

$$a(2d + 2n + 2) - c - n - 1 = 2ad + 2a - c - 1 + (2a - 1)n$$

connective. Since $2a - 1 > 0$, every iterated extension $\mathrm{fib}(X_0 \to X_n)$ is $(2ad + 2a - c - 1)$-connective. By (3.3) and exactness of filtered colimits in Sp, we have $X_0 \simeq \mathrm{colim}_n \mathrm{fib}(X_0 \to X_n)$, whence

$$E(A \oplus M, A) \geq 2ad + 2a - c - 1. \tag{3.10}$$

Now let $A \to B$ be a map of connective k-algebras with $I = \mathrm{fib}(A \to B) \geq d \geq \max\{1, \lceil D/2 \rceil\}$, so that $d \geq 1$ and $2d \geq D$. In (3.4), the third term $E(A \odot_C^B A, B)$ has connectivity at least $a(2d+2) - c \geq 2ad - c$ by (3.5). For the first term $E(C, A)$, the projection $C \to A$ is split with fiber $I \geq d \geq 1$, so we may apply (3.7). Since $\Omega I \geq d - 1 \geq 0$ and $2(d-1) + 2 = 2d \geq D$, the bound (3.10) gives $\Sigma E(A \oplus \Omega I, A) \geq 2ad - c$. Meanwhile $\mathrm{fib}(T_A(I) \to C) \geq 2d \geq D$, so $E(T_A(I), C) \geq 2ad - c$ by hypothesis. From (3.7) we deduce $E(C, A) \geq 2ad - c$, and (3.4) then gives $E(A, B) \geq 2ad - c$. □

Proof of Theorem 3.1. By Lemma 3.2, the hypothesis of Lemma 3.3 holds for $(a, c, D) = (1, D_0, D_0)$. Iterating Lemma 3.3 replaces (a, c, D) by $(2a, c, \max\{1, \lceil D/2 \rceil\})$; after finitely many iterations D reaches 1 and remains 1 thereafter. Hence $E(A, B) \geq 2^m d - D_0$ for all $m \geq 0$ whenever $\mathrm{fib}(A \to B) \geq d \geq 1$, forcing $E(A, B) \simeq 0$. Applying this to the truncation $A \to \pi_0 A$ proves the final claim. □

4. K-theory with compact coefficients

In this section we verify the uniform continuity condition of Theorem 3.1 for K-theory twisted by a compact motive in $\mathrm{Mot}_k^{\mathrm{loc}}$.

4.1. **Corepresentability, duality, and additive products.** Applied to the ring map $f\colon \mathbb{S} \to k$, the base-change adjunction $f_! \dashv f^*$ of Section 2 satisfies $f_!(U_{\mathbb{S}}(\mathbb{S})) = \mathbf{1}$, and for any k-linear category $\mathcal{C}$, $f^*U(\mathcal{C})$ is the absolute motive of its underlying stable category. Since nonconnective K-theory is corepresented by $U_{\mathbb{S}}(\mathbb{S})$ on $\mathrm{Mot}_{\mathbb{S}}^{\mathrm{loc}}$ [3, Theorem 9.8] and f^* preserves colimits, adjunction gives

$$K(X) \simeq \mathrm{Map}_{\mathrm{Mot}_k^{\mathrm{loc}}}(\mathbf{1}, X) \qquad (X \in \mathrm{Mot}_k^{\mathrm{loc}}), \tag{4.1}$$

where K on $\mathrm{Mot}_k^{\mathrm{loc}}$ denotes the unique colimit-preserving extension of K-theory along U. By rigidity of $\mathrm{Mot}_k^{\mathrm{loc}}$, its compact objects coincide with its dualizable objects [7, Theorem 3.1(ii)], and self-duality of $\mathrm{Mot}_k^{\mathrm{loc}}$ is realized by the equivalence

$$\mathrm{Mot}_k^{\mathrm{loc}} \xrightarrow{\sim} \mathrm{Fun}^L(\mathrm{Mot}_k^{\mathrm{loc}}, \mathrm{Sp}), \qquad X \longmapsto K(X \otimes -), \tag{4.2}$$

where Fun^L denotes colimit-preserving functors [7, Theorem 3.1(i)].

Proposition 4.1. *Let $P \in \mathrm{Mot}_k^{\mathrm{loc}}$ be compact. For every $N \in \mathbb{Z}$ there exists an integer d such that for every map $A \to B$ of connective k-algebras,*

$$\mathrm{fib}(A \to B) \geq d \quad \Longrightarrow \quad K\big(P \otimes \mathrm{fib}(U(A) \to U(B))\big) \geq N.$$

The proof relies on the fact that U preserves infinite products [7, Corollary 4.2]. To relate products of categories of perfect modules to products of connective algebras, we use a k-linear version of a result of Córdova Fedeli [8]. Recall that any additive category $\mathcal{A}$ enriched in connective k-modules may be viewed as a k-linear exact category with the split exact structure; we write $\mathrm{Stab}\,\mathcal{A}$ for its Barwick stable envelope [8, Section 2.1], and more generally $\mathrm{Stab}\,\mathcal{E}$ for the stable envelope of any k-linear exact category $\mathcal{E}$. Whenever we apply U to a small stable category, we tacitly pass to its idempotent completion.

Proposition 4.2 (after Córdova Fedeli [8, Proposition 2.10])**.** *Let $(\mathcal{A}_i)_{i\in I}$ be a family of small additive categories enriched in connective k-modules. The canonical functor induces an equivalence*

$$U\Big(\mathrm{Stab}\prod_i \mathcal{A}_i\Big) \xrightarrow{\sim} U\Big(\prod_i \mathrm{Stab}\,\mathcal{A}_i\Big). \tag{4.3}$$

Proof. We apply the second proof of [8, Proposition 2.10] in the k-linear setting. Write $\mathcal{A}_{i,[a,b]}$ for the full subcategory of $\mathrm{Stab}\,\mathcal{A}_i$ of objects admitting a finite filtration with successive cofibers $\Sigma^q A$, where $A \in \mathcal{A}_i$ and $a \leq q \leq b$, and put $\mathcal{A}_{i,[a,\infty)} = \bigcup_{b\geq a} \mathcal{A}_{i,[a,b]}$. These subcategories carry the inherited exact structure [8, Remark 2.1]. The stable envelopes, interval functors, and quotient comparisons in the cited proof are constructed using k-linear presheaves, shifts, finite direct sums, and exact sequences. After idempotent completion their Verdier kernels are closed under $\mathrm{Perf}(k) = \mathrm{thick}(k)$, so these constructions and comparisons take place in $\mathrm{Mot}_k^{\mathrm{loc}}$.

Choose the cofinal family of bounds $f\colon I \to 4\mathbb{Z}_{\geq 0}$, and set $P_0 = \mathrm{Stab}\prod_i \mathcal{A}_{i,[0,\infty)}$ and $P_f = \mathrm{Stab}\prod_i \mathcal{A}_{i,[f(i),\infty)}$. For the inclusion $j\colon P_f \to P_0$ and the equivalence $S = (\Sigma^{f(i)})_i\colon P_0 \xrightarrow{\sim} P_f$, the finite-sum additivity sequences of [8, Lemma 2.9] give $U(jS) = \mathrm{id}$. Thus $U(P_0/P_f) = 0$. The quotient comparison of [8, Proposition 2.10] consequently shows that the top-weight functor

$$(\Sigma^{f(i)})_i\colon\ \mathrm{Stab}\prod_i \mathcal{A}_i \longrightarrow \mathrm{Stab}\prod_i \mathcal{A}_{i,[0,f(i)]}$$

induces an equivalence on U. Since $f(i)/2$ is even, translating by $(\Sigma^{-f(i)/2})_i$ and applying [8, Lemma 2.9] identifies this map on U with the canonical inclusion into $\operatorname{Stab}\prod_i \mathcal{A}_{i,[-f(i)/2,f(i)/2]}$. These stable envelopes have filtered colimit $\prod_i \operatorname{Stab}\mathcal{A}_i$, as in the cited proof. Since U is finitary, the result follows. □

4.2. **Sums, products, and Laurent shifts.**

Lemma 4.3. *Let $f_i\colon A_i \to B_i$ $(i \geq 1)$ be maps of connective k-algebras with $\operatorname{fib}(f_i) \geq d_i$ and $d_i \to +\infty$, and put $V_i = \operatorname{fib}(U(A_i) \to U(B_i))$. Then the canonical map $\bigoplus_i V_i \to \prod_i V_i$ is an equivalence in $\mathrm{Mot}_k^{\mathrm{loc}}$.*

Proof. For a connective k-algebra A, let $\operatorname{Free}(A)$ denote the k-linear additive category of finite free right A-modules, so that the idempotent completion of $\operatorname{Stab}\operatorname{Free}(A)$ is $\operatorname{Perf}(A)$. For each $m \geq 0$, define

$$\mathcal{B}_m = \prod_{i\leq m} \operatorname{Free}(B_i) \times \prod_{i>m} \operatorname{Free}(A_i),$$

with transition functors $\mathcal{B}_m \to \mathcal{B}_{m+1}$ given by extension of scalars along f_{m+1}. Objects of every $\mathcal{B}_m$ and of $\prod_i \operatorname{Free}(B_i)$ are indexed by the same sequences of ranks $(r_i)_{i\geq 1}$. For any two rank sequences (r_i) and (s_i), the mapping k-module in $\mathcal{B}_m$ maps to that in $\prod_i \operatorname{Free}(B_i)$ with fiber $\prod_{i>m} M_{s_i\times r_i}(\operatorname{fib}(f_i))$, which is at least $(\inf_{i>m} d_i)$-connective. Since $d_i \to +\infty$ and all mapping modules are connective, passing to the filtered colimit over m gives an equivalence $\operatorname{colim}_m \mathcal{B}_m \simeq \prod_i \operatorname{Free}(B_i)$.

By Proposition 4.2 and the preservation of infinite products by U [7, Proposition 1.6(i) and Corollary 4.2], the map $U(\operatorname{Stab}\mathcal{B}_0) \to U(\operatorname{Stab}\mathcal{B}_m)$ is identified via the coordinate projections with $\prod_{i\leq m} U(f_i) \times \prod_{i>m} \mathrm{id}_{U(A_i)}$, whose cofiber is $\bigoplus_{i\leq m} \operatorname{cofib} U(f_i)$. Since Stab and U preserve filtered colimits, taking the colimit over m and applying Proposition 4.2 to $\prod_i \operatorname{Free}(B_i)$ shows that the canonical map $\bigoplus_i \operatorname{cofib} U(f_i) \to \prod_i \operatorname{cofib} U(f_i)$ is an equivalence. Desuspending yields $\bigoplus_i V_i \xrightarrow{\sim} \prod_i V_i$. □

Lemma 4.4. *For every discrete k-algebra D, the motive $\Sigma U(D)$ is a retract of $U(D[z, z^{-1}])$, where z is a central variable of degree zero.*

Proof. We recall the relative Morita argument for the Toeplitz algebra from [10, Section 4]. Let $T_D = D\langle x, y\rangle/(yx - 1)$, with x and y central over D, and set $e = 1 - xy$. Then $e^2 = e$ and $ye = ex = 0$, so the elements $x^i e y^j$ $(i, j \geq 0)$ form a system of matrix units over D. They identify the two-sided ideal $I = T_D e T_D$ with the finite-support matrix algebra over D, with quotient $T_D/I \cong D[z, z^{-1}]$ $(x \mapsto z,\ y \mapsto z^{-1})$. Moreover, $eT_De = D \cdot e \cong D$, and multiplication gives $T_De \otimes_D eT_D \xrightarrow{\sim} I$. Since T_De is free over D with basis $\{x^i e\}_{i\geq 0}$ and eT_D is a projective right T_D-module satisfying $eT_D \otimes_{T_D} D[z, z^{-1}] = 0$, the short exact sequence $0 \to I \to T_D \to D[z, z^{-1}] \to 0$ is a projective resolution of $D[z, z^{-1}]$ as a right T_D-module with $I \otimes^{\mathbb{L}}_{T_D} D[z, z^{-1}] = 0$. Hence $T_D \to D[z, z^{-1}]$ is a homological epimorphism. For a right T_D-module X, the fiber of the localization unit $X \to \operatorname{res}(X \otimes^{\mathbb{L}}_{T_D} D[z, z^{-1}])$, taken in Mod_{T_D}, is $X \otimes^{\mathbb{L}}_{T_D} I \simeq Xe \otimes^{\mathbb{L}}_D eT_D$. Thus the kernel of extension of scalars is generated under colimits by the compact module eT_D, whose endomorphism algebra is D. Passing to compact objects gives a localization sequence (up to idempotent completion)

$$\operatorname{Perf}(D) \xrightarrow{-\otimes_D eT_D} \operatorname{Perf}(T_D) \longrightarrow \operatorname{Perf}(D[z, z^{-1}]).$$

Using $yx = 1$ and $ye = ex = 0$, one checks directly that

$$T_D \oplus eT_D \xrightarrow{\sim} T_D, \qquad (a, b) \longmapsto xa + b, \qquad c \longmapsto (yc, ec),$$

are mutually inverse isomorphisms of D-T_D-bimodules. Additivity of U therefore forces $U(-\otimes_D eT_D)\colon U(D)\to U(T_D)$ to be nullhomotopic, so $U(D[z,z^{-1}])\simeq U(T_D)\oplus\Sigma U(D)$. □

Proof of Proposition 4.1. Suppose the proposition fails for P and N. Then for each $i\geq 1$ there exist a map $f_i\colon A_i\to B_i$ of connective k-algebras with $\mathrm{fib}(f_i)\geq i$ and a nonzero class $x_i\in\pi_{n_i}K(P\otimes V_i)$ in degree $n_i<N$, where $V_i=\mathrm{fib}(U(A_i)\to U(B_i))$. Set $L_i=k[s_1^{\pm1},\dots,s_{N-n_i}^{\pm1}]$. By Lemma 4.4 and symmetric monoidality of U, the shifted unit $\Sigma^{N-n_i}\mathbf{1}$ is a retract of $U(L_i)\simeq U(k[z,z^{-1}])^{\otimes(N-n_i)}$, so x_i embeds as a nonzero class $y_i\in\pi_N K(P\otimes V_i\otimes U(L_i))$.

Since L_i is discrete and free over k, the tensored map $f_i\otimes_k L_i\colon A_i\otimes_k L_i\to B_i\otimes_k L_i$ of connective k-algebras still satisfies $\mathrm{fib}(f_i\otimes_k L_i)\geq i$, and symmetric monoidality of U identifies $\mathrm{fib}\,U(f_i\otimes_k L_i)$ with $V_i\otimes U(L_i)$. Applying Lemma 4.3 to the family $(f_i\otimes_k L_i)_{i\geq1}$ gives $\bigoplus_i(V_i\otimes U(L_i))\xrightarrow{\sim}\prod_i(V_i\otimes U(L_i))$ in $\mathrm{Mot}_k^{\mathrm{loc}}$. Now $K(P\otimes-)\simeq\mathrm{Map}_{\mathrm{Mot}_k^{\mathrm{loc}}}(P^\vee,-)$ preserves infinite products (as a representable functor) and preserves infinite direct sums (since $P^\vee$ is compact). Consequently, the canonical map

$$\bigoplus_i K(P\otimes V_i\otimes U(L_i))\longrightarrow\prod_i K(P\otimes V_i\otimes U(L_i))$$

is an equivalence. On π_N, however, the tuple $(y_i)_{i\geq1}$ has infinitely many nonzero coordinates and therefore does not lie in the image of the direct sum, a contradiction. □

5. The cyclotomic trace on localizing motives

In this section we construct the motive J representing the fiber of the mod-p cyclotomic trace and prove that $K(J\otimes P\otimes U(-))$ is truncating on connective k-algebras whenever $P\in\mathrm{Mot}_k^{\mathrm{loc}}$ is compact.

Recall that $\mathrm{TC}(Y)=\mathrm{Map}_{\mathrm{CycSp}}(\mathbb{S}^{\mathrm{triv}},Y)$ for $Y\in\mathrm{CycSp}$ [13]. Topological Hochschild homology defines a finitary localizing invariant into CycSp [7, Section 9.1], which on k-linear categories refines to a symmetric monoidal localizing invariant into $\mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{CycSp})$ [1, Sections 3–4]. We write

$$\mathrm{THH}\colon\mathrm{Mot}_k^{\mathrm{loc}}\longrightarrow\mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{CycSp})$$

for its colimit-preserving symmetric monoidal extension, using the same symbol for its underlying $\mathrm{THH}(k)$-module and underlying spectrum, and write $\mathrm{THH}(A;M)$ for topological Hochschild homology of a k-algebra A with coefficients in an A-bimodule M.

The trivial cyclotomic functor $(-)^{\mathrm{triv}}\colon\mathrm{Sp}\to\mathrm{CycSp}$ is obtained by tensoring the cyclotomic unit with spectra, and is therefore symmetric monoidal. It is left adjoint to TC [13, Proposition IV.4.14].

5.1. TC **modulo** p.

Lemma 5.1. *For every $Y\in\mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{CycSp})$ there is a natural fiber sequence*

$$\Sigma^{-1}(Y/p)_{h\mathbb{T}}\longrightarrow\mathrm{TC}(Y)/p\longrightarrow Y/p.$$

Consequently, $\mathrm{TC}(-)/p$ *preserves colimits on* $\mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{CycSp})$*, and* $\mathrm{TC}(Y)/p\geq b-1$ *whenever* $Y\geq b$.

Proof. By [13, Corollary IV.4.10], the connective cover of $\mathrm{TC}(\mathrm{THH}(\mathbb{F}_p))$ is $H\mathbb{Z}_p$. The symmetric monoidal adjunction $(-)^{\mathrm{triv}}\dashv\mathrm{TC}$ therefore turns the connective-cover map into a map of

commutative algebras $H\mathbb{Z}_p^{\mathrm{triv}} \to \mathrm{THH}(\mathbb{F}_p)$ in CycSp (cf. [13, Corollary IV.4.16]). Composing with $\mathrm{THH}(\mathbb{F}_p) \to \mathrm{THH}(k)$, we may view Y as an $H\mathbb{Z}_p^{\mathrm{triv}}$-module in CycSp. Thus

$$W = Y/p \simeq Y \otimes_{H\mathbb{Z}_p^{\mathrm{triv}}} H\mathbb{F}_p^{\mathrm{triv}}$$

is an $H\mathbb{F}_p^{\mathrm{triv}}$-module in CycSp, and exactness gives $\mathrm{TC}(Y)/p \simeq \mathrm{TC}(W)$. Applying [5, Proposition 2.12] to W gives the natural fiber sequence

$$\mathrm{TC}(W) \longrightarrow W \longrightarrow W_{h\mathbb{T}}.$$

Rotating this sequence proves the assertion. Both W and $W_{h\mathbb{T}}$ preserve colimits in Y, so $\mathrm{TC}(-)/p$ does as well (cf. [5, Corollary 2.13]). If $Y \geq b$, then $W \geq b$ and $W_{h\mathbb{T}} \geq b$, hence $\mathrm{TC}(W) \geq b - 1$. □

5.2. **The trace fiber.** Applying THH to mapping spectra and combining (4.1) with the free-module adjunction over $\mathrm{THH}(k)$ produces a natural transformation

$$K(X) \simeq \mathrm{Map}_{\mathrm{Mot}_k^{\mathrm{loc}}}(\mathbf{1}, X) \longrightarrow \mathrm{Map}_{\mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{CycSp})}(\mathrm{THH}(k), \mathrm{THH}(X)) \simeq \mathrm{TC}(\mathrm{THH}(X)).$$

This agrees with the cyclotomic trace: under the base-change adjunction along $\mathbb{S} \to k$ behind (4.1), restriction to $\mathrm{Mot}_{\mathbb{S}}^{\mathrm{loc}}$ identifies the left-hand side with absolute K-theory via the unit $U_{\mathbb{S}}(\mathbb{S}) \to U_{\mathbb{S}}(k)$, while on the right the free-module adjunction restricts along $\mathbb{S}^{\mathrm{triv}} \to \mathrm{THH}(k)$, recovering the absolute motivic trace that Efimov identifies on connective rings with the classical cyclotomic trace [7, Theorem 9.3 and diagram (9.5)].

By Lemma 5.1 and colimit preservation of THH, the functor $X \mapsto \mathrm{TC}(\mathrm{THH}(X))/p$ preserves colimits on $\mathrm{Mot}_k^{\mathrm{loc}}$. Under the self-duality equivalence (4.2), $K(-)/p$ is represented by $\mathbf{1}/p$, the functor $\mathrm{TC}(\mathrm{THH}(-))/p$ is represented by an object of $\mathrm{Mot}_k^{\mathrm{loc}}$, and the mod-$p$ cyclotomic trace corresponds to a morphism in $\mathrm{Mot}_k^{\mathrm{loc}}$ from $\mathbf{1}/p$ to that representing object. Let $J \in \mathrm{Mot}_k^{\mathrm{loc}}$ be the fiber of this morphism, so that

$$K(J \otimes X) \simeq \mathrm{fib}\big(K(X)/p \longrightarrow \mathrm{TC}(\mathrm{THH}(X))/p\big) \qquad (X \in \mathrm{Mot}_k^{\mathrm{loc}}), \tag{5.1}$$

naturally in X. Writing $\widetilde{U}(A) = \mathrm{fib}(U(A) \to \mathbf{1})$ for an augmented k-algebra $A \to k$, the witness motive of Theorem C is defined by

$$Z = J \otimes \widetilde{U}(k \oplus \Sigma k).$$

5.3. **The derivative of K-theory with coefficients.**

Lemma 5.2. *Let A be a connective k-algebra and M a connective A-bimodule. For $X \in \mathrm{Mot}_k^{\mathrm{loc}}$, there is a natural equivalence*

$$\operatorname*{colim}_{n \geq 0} \Omega^n \,\mathrm{fib}\Big(K\big(X \otimes U(A \oplus \Sigma^n M)\big) \longrightarrow K\big(X \otimes U(A)\big)\Big) \simeq \Sigma\big(\mathrm{THH}(X) \otimes_{\mathrm{THH}(k)} \mathrm{THH}(A; M)\big), \tag{5.2}$$

where the sequential colimit is formed along the suspension assembly maps in the bimodule variable.

Both sides of (5.2) preserve colimits in X. Our strategy is to construct a natural comparison map on motives of small k-linear categories, show that it is an equivalence on proper categories with a compact generator using the theorem of Harpaz, Nikolaus, and Saunier [9], and conclude on all of $\mathrm{Mot}_k^{\mathrm{loc}}$ via Efimov's generation result below. Recall that a k-linear category is *proper* if all of its mapping k-modules are perfect.

Lemma 5.3. *The motives of proper k-linear categories generate $\mathrm{Mot}_k^{\mathrm{loc}}$ as a localizing subcategory.*

Proof. Every small idempotent-complete k-linear category is the filtered colimit of its thick subcategories generated by finitely many objects, each of which has the form $\mathrm{Perf}(C)$ for the endomorphism dg k-algebra C of the direct sum of the generators. Since U is finitary, it suffices to show that $U(C)$ lies in the localizing subcategory generated by proper categories.

Let $\mathcal{B}$ be the upper-triangular dg k-category with objects $\mathbb{N}$, with $\mathcal{B}(i,i) = k$, $\mathcal{B}(i,j) = C$ for $i < j$, and $\mathcal{B}(i,j) = 0$ for $i > j$, and composition induced by multiplication in C. Write $h_i = \mathcal{B}(-,i)$; the unit of C gives $h_i \to h_{i+1}$. By [7, Proposition 3.13(i)], sending each h_i to C defines a localization $\mathrm{Perf}(\mathcal{B}) \to \mathrm{Perf}(C)$, up to idempotent completion, whose kernel is generated as a thick subcategory by the fibers $\mathrm{fib}(h_i \to h_{i+1})$.

Now let (N, V) range over the filtered poset of pairs consisting of an integer $N \geq 0$ and a finite-dimensional dg subcomplex $V \subset C$ containing 1_C. Let $\mathcal{B}_{N,V} \subset \mathcal{B}$ be the dg subcategory on objects $0, \dots, N$ whose morphism complex from i to $j > i$ is the k-linear span of products in C of at most $j - i$ elements of V, and k for $i = j$. The Leibniz rule makes this a dg subcategory with finite-dimensional morphism complexes, so $\mathrm{Perf}(\mathcal{B}_{N,V})$ is proper. Define its representable modules and the full thick subcategory

$$h_i^{N,V} = \mathcal{B}_{N,V}(-,i), \qquad \mathcal{A}_{N,V} = \mathrm{thick}\{\mathrm{fib}(h_i^{N,V} \to h_{i+1}^{N,V}) : 0 \leq i < N\},$$

where the arrows correspond to 1_C. In particular, $\mathcal{A}_{N,V}$ is proper. For $N \leq N'$ and $V \subseteq V'$, extension of scalars along $\mathcal{B}_{N,V} \to \mathcal{B}_{N',V'}$ sends $h_i^{N,V}$ to $h_i^{N',V'}$ and preserves these arrows and their fibers, giving transition functors on $\mathcal{A}_{N,V}$.

Taking filtered colimits in small idempotent-complete stable categories, we obtain

$$\operatorname*{colim}_{N,V} \mathrm{Perf}(\mathcal{B}_{N,V}) \simeq \mathrm{Perf}(\mathcal{B}).$$

Every object index occurs eventually and every mapping complex of $\mathcal{B}$ is the filtered union of its finite-stage mapping complexes. The induced functor $\mathrm{colim}_{N,V} \mathcal{A}_{N,V} \to \mathrm{Perf}(\mathcal{B})$ is fully faithful as well. Indeed, mappings between finite cell modules are computed by finite limits and shifts of the defining mapping complexes, which commute with filtered colimits; the assertion then passes to retracts. Its essential image is the thick subcategory generated by all $\mathrm{fib}(h_i \to h_{i+1})$, since extension of scalars sends the finite-stage representables to h_i. Thus it is precisely the kernel of $\mathrm{Perf}(\mathcal{B}) \to \mathrm{Perf}(C)$. Since U is finitary and localizing,

$$U(C) \simeq \mathrm{cofib}\Big(\operatorname*{colim}_{N,V} U(\mathcal{A}_{N,V}) \longrightarrow \operatorname*{colim}_{N,V} U(\mathrm{Perf}(\mathcal{B}_{N,V}))\Big). \qquad \square$$

We now recall the laced K-theory theorem of [9]. Given a small stable category $\mathcal{P}$ and an exact endofunctor F of $\mathrm{Ind}(\mathcal{P})$, a *laced object* is a pair (V, η) consisting of $V \in \mathcal{P}$ and a morphism $\eta\colon V \to FV$ in $\mathrm{Ind}(\mathcal{P})$. Morphisms from (V,η) to (W,θ) have mapping spectrum

$$\mathrm{fib}\Big(\mathrm{Map}_{\mathcal{P}}(V,W) \xrightarrow{f \mapsto \theta f - F(f)\eta} \mathrm{Map}_{\mathrm{Ind}(\mathcal{P})}(V, FW)\Big).$$

Laced objects form a small stable category, and equipping objects with the zero lacing splits the forgetful functor to $\mathcal{P}$; write $\widetilde{K}^{\mathrm{lace}}(\mathcal{P}, F)$ for the fiber of the induced projection on connective K-theory, and write B_F for the $\mathcal{P}$-bimodule $(V,W) \mapsto \mathrm{Map}_{\mathrm{Ind}(\mathcal{P})}(V, FW)$.

Theorem 5.4 (Harpaz–Nikolaus–Saunier [9, Theorem 4.45])**.** *The trace map from laced K-theory to* THH *exhibits* $\mathrm{THH}(\mathcal{P}; -)$ *as the first Goodwillie derivative of* $\widetilde{K}^{\mathrm{lace}}(\mathcal{P}, -)$*. In particular, for every exact endofunctor F of* $\mathrm{Ind}(\mathcal{P})$*,*

$$\operatorname*{colim}_{n \geq 0} \Omega^n \widetilde{K}^{\mathrm{lace}}(\mathcal{P}, \Sigma^n F) \simeq \mathrm{THH}(\mathcal{P}; B_F).$$

Proof of Lemma 5.2. Step 1: The comparison map. For a small idempotent-complete k-linear category $\mathcal{P}$, set $\mathcal{P}_A = \mathcal{P} \otimes_k \mathrm{Perf}(A)$ and let $F_n = - \otimes_A \Sigma^{n+1} M$ be the induced exact endofunctor of $\mathrm{Ind}(\mathcal{P}_A)$, acting on the A-factor so that $F_n \simeq \Sigma^n F_0$. Given $\widetilde{V} \in \mathcal{P} \otimes_k \mathrm{Perf}(A \oplus \Sigma^n M)$, let $V \in \mathcal{P}_A$ be its extension of scalars along the augmentation $A \oplus \Sigma^n M \to A$, and $\widetilde{V}|_A \in \mathrm{Ind}(\mathcal{P}_A)$ its restriction along the unit $A \to A \oplus \Sigma^n M$. Tensoring with the canonical fiber sequence $\Sigma^n M \to A \oplus \Sigma^n M \to A$ of $(A \oplus \Sigma^n M)$-A-bimodules yields an exact triangle

$$V \otimes_A \Sigma^n M \longrightarrow \widetilde{V}|_A \longrightarrow V$$

in $\mathrm{Ind}(\mathcal{P}_A)$, whose boundary map $V \to V \otimes_A \Sigma^{n+1} M = F_n V$ defines a lacing of V. Because this construction acts purely in the algebra factor, it is natural in exact k-linear functors of $\mathcal{P}$ and in the bimodule, and sends the zero bimodule to the zero lacing. On reduced connective K-theory it therefore induces maps to $\widetilde{K}^{\mathrm{lace}}(\mathcal{P}_A, F_n)$ compatible with the suspension assembly maps in n. Passing to $\mathrm{colim}_{n \geq 0} \Omega^n(-)$ and applying Theorem 5.4 gives a natural map to $\mathrm{THH}(\mathcal{P}_A; B_{F_0})$. Moreover, because $A \oplus \Sigma^n M \to A$ is split, reduced connective K-theory of $\mathcal{P} \otimes_k \mathrm{Perf}(A \oplus \Sigma^n M)$ is the connective cover of the corresponding reduced nonconnective K-theory; since $\Omega^n \tau_{\geq 0} \simeq \tau_{\geq -n} \Omega^n$, the connective cover induces a natural equivalence upon passing to $\mathrm{colim}_{n \geq 0} \Omega^n(-)$. Inverting this equivalence yields a natural transformation from the left-hand side of (5.2) for $X = U(\mathcal{P})$ to $\mathrm{THH}(\mathcal{P}_A; B_{F_0})$.

Step 2: Identification of the target. We show that naturally in $\mathcal{P}$,

$$\mathrm{THH}(\mathcal{P}_A; B_{F_0}) \simeq \Sigma\big(\mathrm{THH}(\mathcal{P}) \otimes_{\mathrm{THH}(k)} \mathrm{THH}(A; M)\big).$$

Choose a k-enriched model for $\mathcal{P}$ and the one-object k-algebra model for A. Objects of the form $x \otimes_k A$ ($x \in \mathcal{P}$) generate $\mathcal{P}_A$ as a thick subcategory, and compactness of x in $\mathrm{Ind}(\mathcal{P})$ identifies $B_{F_0}(x \otimes_k A, y \otimes_k A)$ with $\mathcal{P}(x, y) \otimes_k \Sigma M$. Since a bimodule that is exact in each variable is determined by its restriction to compact generators, $\mathrm{THH}(\mathcal{P}_A; B_{F_0})$ is computed by the cyclic bar construction on these generators. In simplicial degree q, the bar term of A with coefficients in ΣM is $\Sigma M \otimes_{\mathbb{S}} A^{\otimes_{\mathbb{S}} q}$, while that of $\mathcal{P}$ is the direct sum over $(q+1)$-tuples of objects of the tensor product over $\mathbb{S}$ of the $q+1$ mapping k-modules. Taking their relative tensor product over $k^{\otimes_{\mathbb{S}}(q+1)}$ pairs ΣM and each of the q factors of A over k with the corresponding mapping modules of $\mathcal{P}$, recovering the simplicial bar construction of $\mathcal{P}_A$ with coefficients in B_{F_0}. Because geometric realization over the sifted category Δ^{op} commutes with the two-sided bar construction for relative tensor products, realization produces the claimed equivalence over $\mathrm{THH}(k)$.

Step 3: Equivalence on proper categories and extension to $\mathrm{Mot}_k^{\mathrm{loc}}$. Suppose first that $\mathcal{P} = \mathrm{Perf}(C)$ for a proper dg k-algebra C. Then C is perfect over k, hence $C \geq b$ for some $b \in \mathbb{Z}$. On the generator $C \otimes_k A$ of $\mathcal{P}_A$, we have $F_0^q(C \otimes_k A) \simeq \Sigma^q C \otimes_k M^{\otimes_A q} \geq q + b$ since A and M are connective. By finite cell induction and retracts, for every $V \in \mathcal{P}_A$ there is an integer b_V such that $F_n^q(V) \geq (n+1)q + b_V$ for all $q \geq 0$, so the sequential colimit of iterates of any lacing of V vanishes. Applying [17, Propositions 3.2.2 and 3.2.3] (on opposite rings to match our right-module convention) shows that the boundary functor of Step 1 is an equivalence $\mathrm{Perf}(C \otimes_k (A \oplus \Sigma^n M)) \xrightarrow{\sim} \mathrm{Laced}(\mathcal{P}_A, F_n)$ for every $n \geq 0$. Thus the comparison map is an equivalence for $\mathcal{P} = \mathrm{Perf}(C)$ by Theorem 5.4, and hence for every proper k-linear category by passing to filtered colimits of finitely generated thick subcategories.

Finally, both sides of (5.2), evaluated on $X = U(\mathcal{P})$, are finitary localizing invariants of $\mathcal{P}$ (as tensoring with a fixed motive, K, THH, Ω^n, colim_n, and $- \otimes_{\mathrm{THH}(k)} \mathrm{THH}(A; M)$ are all exact and preserve filtered colimits). By the universal property of U, our comparison transformation descends to a natural transformation between colimit-preserving functors on $\mathrm{Mot}_k^{\mathrm{loc}}$. Since it is

an equivalence on the motives of all proper k-linear categories, Lemma 5.3 implies that it is an equivalence on all of $\mathrm{Mot}_k^{\mathrm{loc}}$. □

5.4. **Dundas–Goodwillie–McCarthy with compact coefficients.**

Lemma 5.5. $\mathrm{THH}(J) \simeq 0$ *in* $\mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{CycSp})$.

Proof. Set $a_n = \widetilde{U}(k \oplus \Sigma^n k)$ for $n \geq 0$. Applying Lemma 5.2 with $A = M = k$, desuspending once, and restricting to the cofinal tail $n \geq 1$ gives a natural equivalence

$$\operatorname*{colim}_{n \geq 1} \Sigma^{-n-1} K(X \otimes a_n) \simeq \mathrm{THH}(X) \qquad (X \in \mathrm{Mot}_k^{\mathrm{loc}}).$$

For every $n \geq 1$, the split augmentation $k \oplus \Sigma^n k \to k$ is a map of connective ring spectra inducing an isomorphism on π_0. By Dundas's relative theorem [6, Main Theorem, p. 224], combined with the comparison between classical and Nikolaus–Scholze TC on connective rings [13, Theorems II.4.10–II.4.11 and Corollary III.6.8] and the identification of the motivic trace preceding (5.1), the cyclotomic trace induces an equivalence on relative connective K-theory and relative TC after p-completion, and hence modulo p (as derived p-completion induces an equivalence on mod-p cofibers). Moreover, since $\pi_0(k \oplus \Sigma^n k) = k$, the augmentation induces isomorphisms on K_i for all $i \leq 0$ [3, Theorem 9.53], so relative connective and nonconnective K-theory coincide. Thus

$$K(k \oplus \Sigma^n k, k)/p \xrightarrow{\sim} \mathrm{TC}(k \oplus \Sigma^n k, k)/p \qquad (n \geq 1),$$

which by (5.1) means $K(J \otimes a_n) \simeq 0$ for all $n \geq 1$. Evaluating the stabilization formula above at $X = J$ shows that the underlying spectrum of $\mathrm{THH}(J)$ vanishes. Since the forgetful functor $\mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{CycSp}) \to \mathrm{Sp}$ is conservative [13, Corollary II.1.7], we conclude that $\mathrm{THH}(J) \simeq 0$. □

Proposition 5.6. *Let $P \in \mathrm{Mot}_k^{\mathrm{loc}}$ be compact, and let $A \to B$ be a map of connective k-algebras with $\mathrm{fib}(A \to B) \geq 1$. Then $K(J \otimes P \otimes U(A)) \to K(J \otimes P \otimes U(B))$ is an equivalence.*

Proof. We verify the hypotheses of Theorem 3.1 for the localizing invariant $E(\mathcal{C}) = K(J \otimes P \otimes U(\mathcal{C}))$. For the vanishing derivative, Lemma 5.2 identifies the stabilization of E at (A, M) with $\Sigma(\mathrm{THH}(J \otimes P) \otimes_{\mathrm{THH}(k)} \mathrm{THH}(A; M))$, which vanishes because THH is symmetric monoidal and $\mathrm{THH}(J) \simeq 0$ by Lemma 5.5.

For uniform continuity, (5.1) expresses $E(A, B)$ as the fiber of the mod-p trace from $K(P \otimes \mathrm{fib}(U(A) \to U(B)))/p$ to $\mathrm{TC}(\mathrm{THH}(P) \otimes_{\mathrm{THH}(k)} \mathrm{fib}(\mathrm{THH}(A) \to \mathrm{THH}(B)))/p$. By Proposition 4.1, the connectivity of the source tends to $+\infty$ uniformly with the connectivity of $\mathrm{fib}(A \to B)$. The symmetric monoidal functor THH, followed by the symmetric monoidal forgetful functor to $\mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{Sp})$, sends the dualizable motive P to a perfect underlying $\mathrm{THH}(k)$-module spectrum. Since $\mathrm{THH}(k)$ is connective, $\mathrm{THH}(P) \geq b$ for some $b \in \mathbb{Z}$. If $\mathrm{fib}(A \to B) \geq d \geq 0$, replacing the connective tensor factors one at a time in each simplicial degree of the cyclic bar construction shows that $\mathrm{cofib}(\mathrm{THH}(A) \to \mathrm{THH}(B))$ is a colimit of $(d+1)$-connective spectra, hence $\mathrm{fib}(\mathrm{THH}(A) \to \mathrm{THH}(B)) \geq d$. Tensoring over $\mathrm{THH}(k)$ with $\mathrm{THH}(P)$ yields a $(d+b)$-connective cyclotomic module, so its image under $\mathrm{TC}(-)/p$ is $(d+b-1)$-connective by Lemma 5.1. Thus E is uniformly continuous, and Theorem 3.1 completes the proof. □

Specializing Proposition 5.6 to the augmentation $k \oplus \Sigma k \to k$ immediately yields

$$K(Z \otimes P) \simeq 0 \qquad \text{for every compact } P \in \mathrm{Mot}_k^{\mathrm{loc}}. \tag{5.3}$$

6. The test algebra

Let $k[t]$ be the graded polynomial algebra on a generator of degree -2 with zero differential. In this section we prove that $\pi_{-3}K(Z \otimes U(k[t])) \neq 0$. Because U is symmetric monoidal and $(k \oplus \Sigma k) \otimes_k k[t] \simeq k[t] \oplus \Sigma k[t]$, we have $Z \otimes U(k[t]) \simeq J \otimes \mathrm{fib}(U(k[t] \oplus \Sigma k[t]) \to U(k[t]))$, so (5.1) gives a fiber sequence

$$K(Z \otimes U(k[t])) \longrightarrow K(k[t] \oplus \Sigma k[t], k[t])/p \longrightarrow \mathrm{TC}(k[t] \oplus \Sigma k[t], k[t])/p. \tag{6.1}$$

We show in Proposition 6.2 that the relative K-theory term is connective, and in Proposition 6.3 that the relative $\mathrm{TC}\,/p$ term has nonzero π_{-2}.

6.1. **Relative K-theory.** To control the K-theory of coconnective polynomial algebras, we combine the dévissage theorem of Burklund and Levy [4] with the Bass fundamental exact sequence.

Theorem 6.1 (Polynomial dévissage)**.** *Let C be a commutative regular Noetherian k-algebra and let $B = C[t_1, \dots, t_r]$, where each t_i has negative even degree and the differential is zero. Then extension of scalars induces an equivalence of nonconnective K-theory spectra $K(C) \xrightarrow{\sim} K(B)$.*

Proof. First, B is coconnective with $\pi_0 B = C$, and $\tau_{\leq -1}B$ is a direct sum of negative shifts of C, hence has Tor-amplitude in $[-\infty, -1]$ over C. Since C is regular Noetherian, [4, Theorem 1.1] gives an isomorphism $K_j(C) \xrightarrow{\sim} K_j(B)$ on connective K-theory, i.e., for all $j \geq 0$.

To extend this isomorphism to $j < 0$, let C' range over all rings obtained from C by adjoining finitely many central degree-zero polynomial variables and inverses of such variables. Every such C' is again regular Noetherian, so the connective argument yields $K_0(C') \xrightarrow{\sim} K_0(C'[t_1, \dots, t_r])$ naturally in C'. For any dg k-algebra A and central degree-zero variable z, the natural Bass short exact sequence [18, Theorem 4.2] is

$$0 \longrightarrow K_j(A) \longrightarrow K_j(A[z]) \oplus K_j(A[z^{-1}]) \longrightarrow K_j(A[z, z^{-1}]) \longrightarrow K_{j-1}(A) \longrightarrow 0$$

for every $j \in \mathbb{Z}$. Comparing this sequence along $C' \to C'[t_1, \dots, t_r]$ at $j = 0$ gives an isomorphism on the middle terms and hence on the cokernels K_{-1}; descending induction on j across all C' then shows $K_j(C) \xrightarrow{\sim} K_j(B)$ for all $j < 0$. In particular, since $K_j(C) = 0$ for $j < 0$, $K(B)$ is connective. □

Proposition 6.2. *The spectrum $K(k[t] \oplus \Sigma k[t], k[t])$ is connective.*

Proof. Since $k[t] \oplus \Sigma k[t] \to k[t]$ is split, the relative term is a summand of $K(k[t] \oplus \Sigma k[t])$, so it suffices to show that $K(k[t] \oplus \Sigma k[t])$ is connective.

Consider the localization $\mathrm{Perf}(k[t]) \to \mathrm{Perf}(k[t^{\pm 1}])$ inverting t. On module categories it is the localization away from the compact module $k = \mathrm{cofib}(\Sigma^{-2}k[t] \xrightarrow{t} k[t])$. Its kernel there is $\mathrm{Loc}(k)$, whose compact objects form $\mathrm{thick}(k)$, the kernel of the displayed localization on perfect modules. Using the two-cell free resolution $k[t] \oplus \Sigma^{-1}k[t]$ of k with $\partial(\varepsilon) = t$ ($|\varepsilon| = -1$), the closed $k[t]$-linear endomorphism of homological degree $+1$ sending $\varepsilon \mapsto 1$ and $1 \mapsto 0$ squares strictly to zero and induces an equivalence of k-algebras $k \oplus \Sigma k \xrightarrow{\sim} \mathrm{End}_{k[t]}(k)$. Morita theory therefore identifies the kernel with $\mathrm{Perf}(k \oplus \Sigma k)$. Tensoring this localization sequence over k with $\mathrm{Perf}(k[t])$ produces a localization sequence

$$\mathrm{Perf}(k[t] \oplus \Sigma k[t]) \longrightarrow \mathrm{Perf}(k[t] \otimes_k k[t]) \longrightarrow \mathrm{Perf}(k[t] \otimes_k k[t^{\pm 1}]).$$

By Theorem 6.1, $K(k[t] \otimes_k k[t]) \simeq K(k)$ is connective with $K_0(k[t] \otimes_k k[t]) \cong \mathbb{Z}$ generated by the free rank-one module.

To analyze the third term, set $z = 1 \otimes t$ ($|z| = -2$) and $s = (t \otimes 1)z^{-1}$ ($|s| = 0$), which identifies $k[t] \otimes_k k[t^{\pm 1}]$ with $k[s][z^{\pm 1}]$. The same localization argument for inverting z over the base ring $k[s]$ yields a cofiber sequence

$$K(k[s] \oplus \Sigma k[s]) \longrightarrow K(k[s][z]) \longrightarrow K(k[s][z^{\pm 1}]).$$

Here $k[s] \oplus \Sigma k[s]$ is connective with regular Noetherian $\pi_0 = k[s]$, so $K(k[s] \oplus \Sigma k[s])$ is connective with $K_0 \cong K_0(k[s]) \cong \mathbb{Z}$ [3, Theorem 9.53], while Theorem 6.1 gives $K(k[s][z]) \simeq K(k[s])$, which is also connective with $K_0 \cong \mathbb{Z}$. Under $K_0(k[s] \oplus \Sigma k[s]) \to K_0(k[s][z])$, the generator maps to the class of $k[s] = \mathrm{cofib}(\Sigma^{-2}k[s][z] \xrightarrow{z} k[s][z])$, which vanishes in K_0 because -2 is even so $[\Sigma^{-2}k[s][z]] = [k[s][z]]$. Hence $K(k[s][z^{\pm 1}])$ is connective, and $K_0(k[s][z]) \to K_0(k[s][z^{\pm 1}]) \cong \mathbb{Z}$ is an isomorphism generated by the free rank-one module. Returning to the first localization sequence, the map $K_0(k[t] \otimes_k k[t]) \to K_0(k[t] \otimes_k k[t^{\pm 1}])$ sends the free rank-one module to the free rank-one module and is therefore an isomorphism $\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}$. Since both terms are connective, the long exact sequence shows that $K(k[t] \oplus \Sigma k[t])$ is connective. □

6.2. **Relative** TC**.**

Proposition 6.3. $\pi_{-2}\big(\mathrm{TC}(k[t] \oplus \Sigma k[t], k[t])/p\big) \neq 0$.

Let $k[x]$ denote the graded polynomial algebra on a generator of degree $+2$ with zero differential, and $k[y]$ the discrete polynomial algebra on a generator of degree 0. We prove Proposition 6.3 in three steps, deferring the key cyclotomic splitting to Lemma 6.4.

Step 1: Replacing the square-zero extension by a tensor algebra. Apply Land and Tamme's formula (3.6) to the localizing invariant $E = \mathrm{TC}(-)/p$, the algebra $A = k[t]$, and the diagonal bimodule $I = \Sigma^2 k[t]$. Then $A \oplus \Omega I \simeq k[t] \oplus \Sigma k[t]$ and $T_A(I) \simeq k[t] \otimes_k k[x]$, giving

$$\mathrm{TC}(k[t] \oplus \Sigma k[t], k[t])/p \simeq \Sigma^{-1}\big(\mathrm{TC}(k[t] \otimes_k k[x], k[t])/p\big). \tag{6.2}$$

Both $k[t]$ and $k[x]$ are base changes from $\mathbb{S}$ to k of the free $\mathbb{E}_1$-rings $\mathbb{S}[t]$ and $\mathbb{S}[x]$ on S^{-2} and S^2. By symmetric monoidality of THH, we have $\mathrm{THH}(k[t] \otimes_k k[x]) \simeq \mathrm{THH}(k) \otimes \mathrm{THH}(\mathbb{S}[t]) \otimes \mathrm{THH}(\mathbb{S}[x])$ in CycSp, and $\mathrm{fib}(\mathrm{THH}(k[t] \otimes_k k[x]) \to \mathrm{THH}(k[t]))$ is the direct sum of the biweight pieces (i, j) with $i \geq 0$ and $j \geq 1$, counting occurrences of t and x.

Step 2: Splitting off the diagonal. Let Y_Δ denote the direct sum of the diagonal biweights (n, n) for $n \geq 1$. In Lemma 6.4 below we prove that Y_Δ splits off from $\mathrm{fib}(\mathrm{THH}(k[t] \otimes_k k[x]) \to \mathrm{THH}(k[t]))$ in CycSp, and that there is an equivalence in CycSp

$$Y_\Delta \simeq \mathrm{fib}\big(\mathrm{THH}(k[y]) \to \mathrm{THH}(k)\big) \otimes (\Sigma^\infty_+ \mathbb{T})^{\mathrm{triv}}, \tag{6.3}$$

Choosing a basepoint of $\mathbb{T}$ splits $\mathbb{S}$ off $\Sigma^\infty_+ \mathbb{T}$ in Sp, hence splits $\mathbb{S}^{\mathrm{triv}}$ off $(\Sigma^\infty_+ \mathbb{T})^{\mathrm{triv}}$ in CycSp. Exactness of TC and (6.2) then exhibit

$$\Sigma^{-1}\,\mathrm{fib}\big(\mathrm{TC}(k[y])/p \longrightarrow \mathrm{TC}(k)/p\big)$$

as a direct summand of $\mathrm{TC}(k[t] \oplus \Sigma k[t], k[t])/p$.

Step 3: Artin–Schreier classes. It remains to show that $\pi_{-1}\,\mathrm{fib}(\mathrm{TC}(k[y])/p \to \mathrm{TC}(k)/p) \neq 0$. Let R be either of the discrete commutative $\mathbb{F}_p$-algebras $k[y]$ or k. Then $\mathrm{THH}(R)$ is connective in the cyclotomic t-structure [2, Example 2.3], so its p-typical $\mathrm{TR}(R)$ is connective [2, Theorem 3.21] (using Lemma A.1 for the commutation of homotopy orbits and Tate constructions with the products and uniformly bounded-below sequential limits in that construction), and $\pi_0\,\mathrm{TR}(R) \cong W(R)$ carrying Fr to the Witt vector Frobenius [2, Theorem 6.1]. Since primes $\ell \neq p$ act invertibly on $\mathrm{THH}(R)$, its ℓ-Tate targets vanish, so full $\mathrm{TC}(R)$ agrees with the p-typical equalizer [13, Proposition II.1.9(i)] (cf. [2, Remark 2.5]), which after p-completion—and

hence modulo p—is equivalent to $\mathrm{fib}(1 - \mathrm{Fr}\colon \mathrm{TR}(R) \to \mathrm{TR}(R))$ [2, Warning 2.6]. Because $\mathrm{TR}(R) \geq 0$, the fiber of $1 - \mathrm{Fr}$ vanishes below degree -1, and therefore

$$\pi_{-1}(\mathrm{TC}(R)/p) \cong \mathrm{coker}(1 - \mathrm{Fr}\colon W(R) \to W(R))/p.$$

Projection to the initial Witt coordinate $W(R) \twoheadrightarrow R$ sends $\mathrm{Fr}(a)$ to a_0^p and annihilates $pW(R)$ since $\mathrm{char}(R) = p$, yielding a natural surjection

$$\pi_{-1}(\mathrm{TC}(R)/p) \twoheadrightarrow R/\{f^p - f : f \in R\}.$$

Passing to reduced terms along the split augmentation $k[y] \to k$ $(y \mapsto 0)$ gives a surjection from $\pi_{-1}\,\mathrm{fib}(\mathrm{TC}(k[y])/p \to \mathrm{TC}(k)/p)$ onto $yk[y]/\{f^p - f : f \in yk[y]\}$. In this quotient the class of y is nonzero, because for every nonzero $f \in yk[y]$ we have $\deg(f^p - f) = p \deg f \geq p > 1 = \deg y$. Thus $\pi_{-1}\,\mathrm{fib}(\mathrm{TC}(k[y])/p \to \mathrm{TC}(k)/p) \neq 0$, completing the proof of Proposition 6.3 modulo Lemma 6.4.

6.3. **The diagonal of the cyclic bar construction.**

Lemma 6.4. *The diagonal sum Y_Δ is a direct summand of* $\mathrm{fib}(\mathrm{THH}(k[t] \otimes_k k[x]) \to \mathrm{THH}(k[t]))$ *in* CycSp*, and there is an equivalence* (6.3) *in* CycSp*.*

Geometrically, in biweight (n, n) the representation spheres $S^{-2\rho_n}$ from t and $S^{2\rho_n}$ from x cancel by duality once extended to circle representations; what remains is the diagonal circle action on $\mathbb{T}/C_n \times \mathbb{T}/C_n$, which shears into $\mathbb{T}/C_n$ (modeling a degree-zero variable y) times a circle with trivial action. We check that every step of this identification commutes with the cyclotomic Frobenius.

Proof. Step 1: Induced models. For $m \in \{-2, 0, 2\}$ and $n \geq 1$, the weight-n summand of the cyclic bar construction $\mathrm{THH}(\mathbb{S}[S^m])$ of the free $\mathbb{E}_1$-ring on S^m admits the $\mathbb{T}$-equivariant model

$$\mathbb{T}_+ \wedge_{C_n} \big((\Delta^{n-1})_+ \wedge S^{m\rho_n}\big),$$

where $\rho_n = \mathbb{R}[C_n]$ is the real regular representation of C_n. Indeed, a simplex of weight n distributes a cyclic word of n letters across its slots and is generated from the nondegenerate $(n-1)$-simplex with one letter per slot by merging adjacent slots and inserting units, modulo cyclic rotation of the n letters. In geometric realization, choose ordered lifts $u_0 \leq u_1 \leq \cdots \leq u_{n-1} \leq u_0 + 1$ in $\mathbb{R}$ of the n marked points on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, put $u_n = u_0 + 1$, take the gaps $u_{i+1} - u_i$ as barycentric coordinates on Δ^{n-1}, and set

$$c_n = \frac{1}{n} \sum_{i=0}^{n-1} u_i - \frac{n-1}{2n} \pmod{\mathbb{Z}}.$$

Cyclic rotation replaces $(u_0, \ldots, u_{n-1})$ by $(u_1, \ldots, u_{n-1}, u_0 + 1)$, which adds $1/n$ to c_n and cyclically permutes the gaps and the n smash factors of S^m, yielding the induced space above. Contracting Δ^{n-1} linearly to its barycenter gives $\mathbb{T}_+ \wedge_{C_n} S^{m\rho_n}$, following the left induction convention of [15, Proposition 4.13].

Step 2: Frobenius on weights. For any prime p, the p-cyclotomic Frobenius on the cyclic bar construction is defined via the levelwise C_p-Tate diagonal into the p-fold edgewise subdivision [13, Definition III.2.3, Corollary III.3.8, and Appendix B]. On each simplex, naturality of the Tate diagonal with respect to the inclusion of a word summand sends a word of weight n into the C_p-Tate construction of its p-fold concatenation in weight pn, compatibly with cyclic structure maps. Realizing and mapping into the Tate construction of the total realization shows that the Frobenius sends biweight (i, j) into the C_p-Tate construction of biweight (pi, pj). Because both the diagonal $\{(n, n) : n \geq 1\}$ and its complement in $\{(i, j) : i \geq 0,\, j \geq 1\}$ are preserved under

$(i,j) \mapsto (pi, pj)$ and $(-)^{tC_p}$ preserves binary direct sums, Y_Δ splits off as a direct summand in CycSp.

On the sphere and configuration factors, the Tate diagonal factors as geometric fixed points of the genuine C_p-norm followed by the Borel-to-Tate comparison [13, Remark III.1.5 and Corollary III.6.8]. On $\mathbb{R}[C_n]$, p-fold concatenation corresponds to normalized block repetition

$$\mathbb{R}[C_n] \xrightarrow{\sim} \mathbb{R}[C_{pn}]^{C_p}, \qquad (v_k)_{0\le k<n} \longmapsto \left(\frac{v_k}{\sqrt{p}}\right)_{k+rn} \qquad (0 \le r < p).$$

On inverse representation spheres we use the Spanier–Whitehead dual of this isometry's inverse. On the configuration coordinates, a lift of the p-fold repeated configuration has marked points $(u_k + r)/p$ $(0 \le k < n,\ 0 \le r < p)$, whose gaps repeat the old gaps divided by p and whose circle coordinate is identically $c_{pn} = c_n/p$. In particular, p-fold repetition commutes strictly with linear contraction of Δ^{n-1} to its barycenter. The coefficient factor $\mathrm{THH}(k)$ contributes via its own Frobenius combined through lax symmetric monoidality of $(-)^{tC_p}$.

Step 3: Cancelling the representation spheres. To cancel the two spheres in biweight (n,n) across distinct orbit factors, we extend the C_n-representations to $\mathbb{T}$. Let $\lambda_n = \bigoplus_{j=0}^{n-1} \mathbb{C}(j)$, where $z \in \mathbb{T} \subset \mathbb{C}^\times$ acts on $\mathbb{C}(j)$ by z^j. Writing $\zeta_n = e^{2\pi i/n}$, the unitary discrete Fourier transform

$$\mathbb{C}[C_n] \xrightarrow{\sim} \lambda_n|_{C_n}, \qquad (v_k)_{0\le k<n} \longmapsto \left(\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \zeta_n^{-jk} v_k\right)_{0\le j<n}$$

is an isometry on underlying real inner product spaces $2\rho_n \xrightarrow{\sim} \lambda_n|_{C_n}$. Under normalized block repetition, summing powers of ζ_p annihilates characters not divisible by p and identifies the (pj)-th Fourier coefficient with the old j-th coefficient, intertwining block repetition with the fixed-point identification $\lambda_n \xrightarrow{\sim} (\lambda_{pn})^{C_p}$ along the residual circle isomorphism $\mathbb{T}/C_p \xrightarrow{\sim} \mathbb{T}$ $(w \mapsto w^p)$.

Regard $E = S^{\lambda_{pn}}$ as a genuine $\mathbb{T}$-spectrum with respect to the family of finite subgroups. The symmetric monoidal functor Φ^{C_p} takes values in genuine $\mathbb{T}/C_p$-spectra. After passage to the underlying spectra with residual $\mathbb{T}/C_p$-action, the Borel-completion unit gives a natural lax symmetric monoidal comparison

$$c_E \colon \Phi^{C_p} E \longrightarrow (uE)^{tC_p},$$

where uE retains its $\mathbb{T}$-action before the Tate construction [13, discussion following Definition II.2.15, Construction III.6.2, and (T5) preceding Theorem III.6.7]. The representation sphere has dual $E^\vee = S^{-\lambda_{pn}}$, and the fixed-subspace isometries identify $\Phi^{C_p} E \simeq S^{\lambda_n}$ and $\Phi^{C_p}(E^\vee) \simeq S^{-\lambda_n}$ with their residual actions. Applying this residual-equivariant comparison to $\mathrm{ev} \colon E^\vee \otimes E \xrightarrow{\sim} \mathbb{S}$ gives a commutative square in $\mathrm{Sp}^{B(\mathbb{T}/C_p)}$.

$$\begin{array}{ccc} S^{-\lambda_n} \otimes S^{\lambda_n} \simeq \Phi^{C_p}(E^\vee) \otimes \Phi^{C_p}(E) & \xrightarrow[\sim]{\Phi^{C_p}(\mathrm{ev})} & \mathbb{S} \\ \downarrow{\scriptstyle c_{E^\vee} \otimes c_E} & & \downarrow{\scriptstyle c_{\mathbb{S}}} \\ (uE^\vee)^{tC_p} \otimes (uE)^{tC_p} & \xrightarrow[\mathrm{ev}^{tC_p} \circ \mu]{} & \mathbb{S}^{tC_p}. \end{array}$$

For the moving directions, write $\lambda_{pn} = (\lambda_{pn})^{C_p} \oplus V$. Since p is prime, the unit sphere $S(V)$ is a finite free C_p-CW complex. The sequence $S(V)_+ \to \mathbb{S} \xrightarrow{a_V} S^V$ therefore makes the Euler map a_V an equivalence after either Φ^{C_p} or the Tate construction, also after tensoring with any coefficient spectrum. The comparisons for the positive sphere and its dual use this map and its inverse,

which cancel under the displayed evaluation pairing. Now untwist the induced bundles via the $\mathbb{T}$-equivariant homeomorphism $[u,v]\mapsto([u],u\cdot v)$, which gives $\mathbb{T}_+\wedge_{C_n}S^{\pm 2\rho_n}\simeq S^{\pm\lambda_n}\wedge(\mathbb{T}/C_n)_+$; since a lift u_{new} with $u_{\mathrm{new}}^p=u$ acts on $\mathbb{C}(pj)$ by $u_{\mathrm{new}}^{pj}=u^j$, this untwisting commutes strictly with p-fold repetition. Applying the evaluation square above cancels $S^{-\lambda_n}$ against S^{λ_n} to the Tate unit $c_{\mathbb{S}}$, which disappears into the remaining factors by lax unitality. Because we work with underlying real isometries of $\mathbb{C}[C_n]$, this holds equally for $p=2$.

Step 4: Shearing the orbit factors. After cancelling the spheres, biweight (n,n) of Y_Δ is $\mathrm{THH}(k)\otimes\Sigma^\infty_+(\mathbb{T}/C_n\times\mathbb{T}/C_n)$. The $\mathbb{T}$-equivariant homeomorphism

$$\mathbb{T}/C_n\times\mathbb{T}/C_n\xrightarrow{\sim}\mathbb{T}/C_n\times\mathbb{T}^{\mathrm{triv}},\qquad([u],[u'])\longmapsto([u],(u'/u)^n),$$

endows the second factor with the trivial circle action. Moreover, if $w^p=u$ and $(w')^p=u'$, then $(w'/w)^{pn}=(u'/u)^n$, so p-fold repetition is the identity on $\mathbb{T}^{\mathrm{triv}}$ (whose Borel-to-Tate map $c_{\Sigma^\infty_+\mathbb{T}}$ is the cyclotomic Frobenius of $(\Sigma^\infty_+\mathbb{T})^{\mathrm{triv}}$), while the first factor $\mathbb{T}/C_n$ carries precisely the weight-n Frobenius of $\mathrm{THH}(\mathbb{S}[y])$ with $|y|=0$. Summing over $n\geq 1$ yields the cyclotomic equivalence (6.3). □

6.4. **Proofs of Theorems C and B.**

Proof of Theorem C. In the fiber sequence (6.1), the middle term $K(k[t]\oplus\Sigma k[t],k[t])/p$ is connective by Proposition 6.2, so the boundary map induces an isomorphism

$$\pi_{-3}K\big(Z\otimes U(k[t])\big)\cong\pi_{-2}\big(\mathrm{TC}(k[t]\oplus\Sigma k[t],k[t])/p\big),$$

which is nonzero by Proposition 6.3. On the other hand, for every compact $P\in\mathrm{Mot}_k^{\mathrm{loc}}$, we proved $K(Z\otimes P)\simeq 0$ in (5.3); since $P^\vee$ is also compact, (4.1) gives $\mathrm{Map}_{\mathrm{Mot}_k^{\mathrm{loc}}}(P,Z)\simeq K(P^\vee\otimes Z)\simeq 0$. Thus Z is nonzero and right orthogonal to all compact objects, so $\mathrm{Mot}_k^{\mathrm{loc}}$ is not compactly generated. □

Proof of Theorem B in positive characteristic. Let κ be a field of characteristic $p>0$. When κ is perfect, the colimit-preserving functor $K(Z\otimes-)\colon\mathrm{Mot}_\kappa^{\mathrm{loc}}\to\mathrm{Sp}$ of Theorem C vanishes on $(\mathrm{Mot}_\kappa^{\mathrm{loc}})^\omega$, hence on $\mathrm{Loc}((\mathrm{Mot}_\kappa^{\mathrm{loc}})^\omega)$, whereas $K(Z\otimes U_\kappa(\kappa[t]))\not\simeq 0$; thus $U_\kappa(\kappa[t])\notin\mathrm{Loc}((\mathrm{Mot}_\kappa^{\mathrm{loc}})^\omega)$. For a general field κ of characteristic p, let $f\colon\kappa\to\kappa^{\mathrm{perf}}$ be an embedding into its perfect closure. If $U_\kappa(\kappa[t])$ belonged to $\mathrm{Loc}((\mathrm{Mot}_\kappa^{\mathrm{loc}})^\omega)$, then by (2.1) and Lemma 2.1 its base change $f_!U_\kappa(\kappa[t])\simeq U_{\kappa^{\mathrm{perf}}}(\kappa^{\mathrm{perf}}[t])$ would belong to $\mathrm{Loc}((\mathrm{Mot}_{\kappa^{\mathrm{perf}}}^{\mathrm{loc}})^\omega)$, which we have just ruled out. Combined with the characteristic-zero case proved in Section 2, this establishes Theorem B for every field κ, and hence Theorem A. □

Appendix A. Bounded-below inverse limits

The following estimate justifies commuting C_p-homotopy orbits and Tate constructions with uniformly bounded-below limits in the construction of TR (specifically products, where $d=0$, and sequential inverse limits, where $d=1$).

Lemma A.1. *Let I be a small category and let $F\colon I\to\mathrm{Sp}^{BC_p}$ satisfy $F(i)\geq N$ for every $i\in I$. Suppose that $\lim_I\colon\mathrm{Fun}(I,\mathrm{Sp})\to\mathrm{Sp}$ sends connective diagrams to $(-d)$-connective spectra for some integer $d\geq 0$. Then the canonical assembly maps*

$$(\lim_I F)_{hC_p}\longrightarrow\lim_I F(i)_{hC_p},\qquad(\lim_I F)^{tC_p}\longrightarrow\lim_I F(i)^{tC_p}$$

are equivalences.

Proof. Choose a free contractible C_p-CW complex EC_p with finite m-skeleta E_m, and define $P_m(Y) = (E_m)_+ \otimes_{C_p} Y$ for $Y \in \mathrm{Sp}^{BC_p}$. Since E_m is built from finitely many free C_p-cells, P_m is formed from finitely many finite colimits of the forgetful functor $\mathrm{Sp}^{BC_p} \to \mathrm{Sp}$, and therefore preserves all limits. Moreover, for any $Y \geq a$, cellular filtration gives

$$\mathrm{cofib}\big(P_m(Y) \longrightarrow Y_{hC_p}\big) \geq a + m + 1.$$

Put $X = \lim_I F$, so $X \geq N - d$. Comparing $P_m(X) \xrightarrow{\sim} \lim_I P_m(F(i))$ with $X_{hC_p} \to \lim_I F(i)_{hC_p}$, both vertical cofibers are at least $(N + m + 1 - d)$-connective. Letting $m \to +\infty$ shows that $X_{hC_p} \to \lim_I F(i)_{hC_p}$ is an equivalence, and the C_p-norm sequence together with limit preservation of $(-)^{hC_p}$ yields the Tate equivalence. $\square$

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