THE FAUSK–LEWIS–MAY CONJECTURE FOR COMPACT LIE GROUPS

Abstract. For every compact Lie group G, we prove that the canonical homomorphism $\rho_G\colon V(G) \to V'(G)$ from the Grothendieck group of homotopy representations to that of generalized homotopy representations is an isomorphism. This settles Conjecture 4.5 of Fausk, Lewis, and May and identifies $V(G)$ with the Picard group $\mathrm{Pic}(\mathrm{Sp}^G)$ of the symmetric monoidal ∞-category of genuine G-spectra. Together with earlier work of Fausk–Lewis–May and Bauer, this yields a canonical short exact sequence

$$0 \longrightarrow \mathrm{Pic}(A(G)) \longrightarrow \mathrm{Pic}(\mathrm{Sp}^G) \longrightarrow D(G) \longrightarrow 0$$

for every compact Lie group G, where $A(G)$ is the Burnside ring and $D(G)$ is Bauer's group of dimension functions. At the unstable level, we show that for every generalized homotopy representation Y of G there exist a finite-dimensional real G-representation V and a homotopy representation X equivalent to $Y \wedge S^V$ in $\mathcal{S}_*^G$.

1. Introduction

Let G be a compact Lie group, and write Sp^G for the symmetric monoidal ∞-category of genuine G-spectra. Its Picard group $\mathrm{Pic}(\mathrm{Sp}^G)$ of $\wedge$-invertible objects is closely tied to unstable equivariant homotopy spheres [3]. Let $\mathcal{O}_G$ denote the topological orbit ∞-category of G, with objects G/H for closed subgroups $H \leq G$, and let $\mathcal{S}_*^G \simeq \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{S}_*)$ be the symmetric monoidal ∞-category of pointed genuine G-spaces under pointwise smash product $\wedge$ [2, 4]. An object $Y \in \mathcal{S}_*^G$ is *compact*, written $Y \in (\mathcal{S}_*^G)^\omega$, if $\mathrm{Map}_{\mathcal{S}_*^G}(Y, -)$ preserves filtered colimits, or equivalently if Y is a retract in $\mathcal{S}_*^G$ of a finite based G-CW complex. A *generalized homotopy representation* of G is a compact object $Y \in (\mathcal{S}_*^G)^\omega$ such that $Y^H = \mathrm{ev}_{G/H}(Y) \in \mathcal{S}_*$ is equivalent to a sphere $S^{n_Y(H)}$ of dimension $n_Y(H) \geq 0$ for every closed subgroup $H \leq G$ [3, Definition 0.4].

A *homotopy representation* of G is a generalized homotopy representation that admits a based G-CW model X with fixed-point covering dimensions

$$\dim X^H = n_X(H) \qquad \text{for every closed subgroup } H \leq G$$

[3, §4]. Up to unreduced suspension, this agrees with the notion introduced by tom Dieck and Petrie for finite groups [12] and studied by Bauer for compact Lie groups [1]. The equality concerns the covering dimension of the chosen cellular model. Some generalized homotopy representations have no such model even after suspension by a trivial representation. For any odd prime p, unreduced suspension of [12, Example 1.4] yields a finite based C_p-CW complex Y with $Y^{C_p} \simeq S^n$ and $Y \simeq S^n$ such that the inclusion $Y^{C_p} \to Y$ has degree $j > 1$ coprime to p. If $Y \wedge S^N$ $(N \geq 0)$ admitted a based C_p-CW model X of dimension $N' = n + N$, the pair sequence in degree N' would embed $\mathbb{Z}/j\mathbb{Z}$ into $H_{N'}(X, X^{C_p}; \mathbb{Z})$. Since $N' = \dim X$, this relative group is a subgroup of the top relative cellular chain group and is therefore free abelian. This rules out such a model.

Group completion inverts all representation spheres S^V. Equivalence classes in $\mathcal{S}_*^G$ of generalized homotopy representations form a commutative monoid under $\wedge$ that contains those of homotopy representations as a submonoid (Lemma 2.2). Let $V'(G)$ and $V(G)$ denote their Grothendieck groups [3, Definition 4.3]. Every object of $\mathrm{Pic}(\mathrm{Sp}^G)$ has the form $\Sigma^{-V}\Sigma_G^\infty Y$ for a real G-representation V and a generalized homotopy representation Y, and

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suspension induces a canonical isomorphism $\sigma_G\colon V'(G) \xrightarrow{\cong} \mathrm{Pic}(\mathrm{Sp}^G)$ [3, Theorem 0.5 and Proposition 4.6]. By contrast, Bauer's algebraic description of dimension functions applies to $V(G)$ rather than $V'(G)$ [1]. Fausk, Lewis, and May conjectured that the comparison homomorphism $\rho_G\colon V(G) \to V'(G)$ is an isomorphism for every compact Lie group G [3, Conjecture 4.5]. We prove this conjecture in full generality.

Theorem 1.1. *For every compact Lie group G, both homomorphisms in the composite*

$$V(G) \xrightarrow{\rho_G} V'(G) \xrightarrow{\sigma_G} \mathrm{Pic}(\mathrm{Sp}^G)$$

are isomorphisms, where ρ_G is the canonical comparison map and σ_G is the suspension-spectrum isomorphism of [3, Proposition 4.6].

Write $S^V \in (\mathcal{S}_*^G)^\omega$ for the one-point compactification of a finite-dimensional real orthogonal G-representation V, pointed at infinity. Theorem 1.1 reduces to the following unstable stabilization theorem.

Theorem 1.2. *Let G be a compact Lie group. For every generalized homotopy representation $Y \in (\mathcal{S}_*^G)^\omega$, there exist a finite-dimensional real G-representation V and a homotopy representation X of G with $X \simeq Y \wedge S^V$ in $\mathcal{S}_*^G$.*

Let $C(G)$ be the additive group of continuous integer-valued functions on the space of conjugacy classes of closed subgroups of G in the Hausdorff-metric topology, and let $D(G) \subseteq C(G)$ be Bauer's subgroup of dimension functions [1, Definition 2.1]. Together with [3, Theorems 0.1, 0.2, and 3.2] and [1, Theorem 2.4], Theorem 1.1 yields the short exact sequence conditionally stated at the end of [3, §4].

Corollary 1.3. *For every compact Lie group G, there is a canonical short exact sequence of abelian groups*

$$0 \longrightarrow \mathrm{Pic}(A(G)) \xrightarrow{c} \mathrm{Pic}(\mathrm{Sp}^G) \xrightarrow{d} D(G) \longrightarrow 0,$$

where c sends invertible modules over the Burnside ring $A(G)$ to retracts of finite wedges of the genuine sphere spectrum [3, Theorem 0.2]*, and $d([E])(H)$ is the sphere dimension of the geometric fixed-point spectrum $\Phi^H E \in \mathrm{Sp}$.*

For finite groups and tori the stabilization and comparison theorems are already known [3, Proposition 4.2 and Corollary 4.4]. Knutsen proved both theorems for finite groups and for compact Lie groups G for which finitely generated projective $\mathbb{Z}[\pi_0 G]$-modules are stably free [7, Theorems 5.4 and 5.6]. In the positive-dimensional case of Knutsen's attachment lemma, stable freeness allows one to make the bottom homology of a cofiber free by adding finitely many free cells [7, Lemma 3.4].

We use an integral homology calculation to show that this bottom homology is projective and generates the higher homology under the action of the identity component of the Weyl group. A projective module becomes free after adding a suitable free module of possibly infinite rank. The additional cells realizing that free module all have the same disk degree, so their number does not affect the dimension bound. The relative Hurewicz theorem then realizes a free basis by disks. We perform the resulting cell attachments over the orbit types of a single representation suspension of Y.

2. Genuine G-Spaces and Cellular Dimension

Throughout the paper, G denotes a compact Lie group and all subgroups are closed. Let $\mathcal{S}_*$ be the symmetric monoidal ∞-category of pointed spaces under smash product $\wedge$, and let $\mathcal{O}_G$ be the topological orbit ∞-category of G, with objects G/H ($H \leq G$) and mapping spaces $\mathrm{Map}_{\mathcal{O}_G}(G/H, G/K) \simeq (G/K)^H$. By Elmendorf's theorem, the ∞-category of pointed genuine G-spaces is canonically equivalent to

$$\mathcal{S}_*^G \simeq \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{S}_*)$$

under the pointwise smash product $\wedge$ with unit S^0 [2], [4, Appendix C]. Write $(-)^H = \mathrm{ev}_{G/H}\colon \mathcal{S}_*^G \to \mathcal{S}_*$ for ordinary pointed fixed points. Limits, colimits, and smash products in $\mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{S}_*)$ are computed objectwise, so the functors $\{(-)^H\}_{H\le G}$ are jointly conservative, preserve all limits and colimits, and commute with $\wedge$:

$$(A \wedge B)^H \simeq A^H \wedge B^H \qquad \text{in } \mathcal{S}_*. \tag{2.1}$$

The pointed orbit $(G/H)_+ \in \mathcal{S}_*^G$ corepresents $(-)^H$ and is therefore compact. Finite based G-CW complexes are also compact, since their cell attachments use finite colimits of $(G/H)_+ \wedge S_+^{j-1}$ and $(G/H)_+ \wedge D_+^j$. Conversely, every based G-CW complex X is the filtered colimit in $\mathcal{S}_*^G$ of its finite based G-subcomplexes: for each $H \le G$, any compact subset of X^H projects to a compact subset of the ordinary CW orbit space X/G, hence lies in the image of a finite G-subcomplex of X, so the filtered colimit computes all homotopy groups of X^H [4, Appendix C]. For any compact object $Y \in (\mathcal{S}_*^G)^\omega$ presented by a based G-CW complex X, the identity of Y therefore factors in $\mathcal{S}_*^G$ through a finite based G-subcomplex of X. Hence $(\mathcal{S}_*^G)^\omega$ consists of the retracts of finite based G-CW complexes. Moreover, for any $K_1, K_2 \le G$, the diagonal G-manifold $G/K_1 \times G/K_2$ is compact and smooth, hence admits a finite G-CW triangulation [6, Theorem 1.3 and Corollary B]. Thus $(G/K_1)_+ \wedge (G/K_2)_+ \simeq (G/K_1 \times G/K_2)_+$ lies in $(\mathcal{S}_*^G)^\omega$. Since $\wedge$ preserves colimits in each variable, $(\mathcal{S}_*^G)^\omega$ is closed under $\wedge$.

A *based G-CW complex* has a G-fixed basepoint that is a 0-cell. A cell $G/K \times D^j$ has *disk degree* j and *geometric dimension* $j + \dim(G/K)$. We write $X^{(j)}$ for the disk j-skeleton. Attaching maps for cells of disk degree j land in $X^{(j-1)}$. Based G-CW complexes present $\mathcal{S}_*^G$, and equivalences between them are based G-homotopy equivalences [10, p. 210].

For a closed subgroup $H \le G$, write $W_G(H) = N_G(H)/H$ and let $\pi : N_G(H) \to W_G(H)$ be the quotient. The space X^H carries an action of $W_G(H)$, with $(X^H)^L = X^{\pi^{-1}(L)}$ for $L \le W_G(H)$. Each fixed manifold $(G/K)^H$ is a finite disjoint union of homogeneous $W_G(H)$-manifolds. It gives X^H a based $W_G(H)$-CW structure with

$$(X^H)^{(j)} = (X^{(j)})^H \qquad (j \ge 0) \tag{2.2}$$

[5, Lemma 4.3 and Theorem 4.4]. Its covering dimension is

$$\dim X^H = \sup\{\, j + \dim(G/K)^H : G/K \times D^j \text{ is a cell of } X \text{ with } (G/K)^H \neq \varnothing \,\} \tag{2.3}$$

[6, Lemma 1.2]. Thus X is *finite-dimensional* ($\dim X < \infty$) if and only if the disk degrees of its cells are bounded above.

Lemma 2.1. *For any compact Lie group W, every object $Y \in (\mathcal{S}_*^W)^\omega$ is equivalent in $\mathcal{S}_*^W$ to a finite-dimensional based W-CW complex with finitely many orbit types.*

Proof. Represent the retraction by based W-maps $Y \xrightarrow{i} E \xrightarrow{r} Y$ between CW models, with E finite and $ri \simeq \mathrm{id}_Y$. The telescopes of ri and ir are equivalent, since both are formed by taking cofinal subsequences in the alternating telescope of i and r. The former is equivalent to Y. Replace ir by a based cellular map e [9, Theorem (4.4)]. Its reduced telescope is a based W-CW complex of dimension at most $\dim E + 1$ with the same orbit types as E. □

Lemma 2.2. *Equivalence classes of generalized homotopy representations form a commutative submonoid of $\pi_0((\mathcal{S}_*^G)^\omega)$ under $\wedge$ with unit $[S^0]$ that contains the classes of homotopy representations. Moreover, for every finite-dimensional real orthogonal G-representation V, S^V admits a finite based G-CW structure and is a homotopy representation of G.*

Proof. Since $(\mathcal{S}_*^G)^\omega$ is closed under $\wedge$, (2.1) gives the assertion for generalized homotopy representations. For homotopy representations, choose based G-CW models A and B with $\dim A^H = n_A(H)$ and $\dim B^H = n_B(H)$ for all $H \le G$. In the external $(G \times G)$-CW smash product, the $\Delta(H)$-fixed part of a product cell has geometric dimension equal to the sum of the dimensions of its two H-fixed factors. By (2.3), each factor $A^H \simeq S^{n_A(H)}$ has a

non-basepoint cell piece of dimension $n_A(H)$ (also when $n_A(H) = 0$, where A^H has two path components), and likewise for B^H, so $\dim(A^H \wedge B^H) = n_A(H) + n_B(H)$. Illman's relative restriction theorem along $\Delta(G) \leq G \times G$ provides a based G-CW model of $A \wedge B$ with the same fixed-point covering dimensions [6, Theorem A (rel)]. Finally, S^0 consists of two G-fixed 0-cells. When $\dim V > 0$, equivariant triangulation of $(D(V), S(V))$ gives $S^V \cong D(V)/S(V)$ a finite based G-CW structure, and $(S^V)^H = S^{V^H}$ has dimension $\dim V^H$ [6, Theorem 1.3]. □

3. Homology of Free W-Complexes with Bounded Support

Let W be a compact Lie group of dimension $k \geq 0$ with identity component W_0, finite component group $\Gamma = W/W_0 = \pi_0 W$, and group ring $R = \mathbb{Z}[\Gamma]$. Since W_0 is path-connected, the W-action on the integral homology of any object of $\mathcal{S}_*^W$ factors through Γ. Conjugation $c_w(w_0) = ww_0w^{-1}$ by $w \in W$ is homotopic to the identity when $w \in W_0$ and therefore defines a left Γ-action on $H_t(W_0; \mathbb{Z})$.

Lemma 3.1. *Let Γ be a finite group and $R = \mathbb{Z}[\Gamma]$. Any left R-module P that is free as an abelian group and admits a finite projective resolution over R is projective over R.*

Proof. Since Γ is finite, the group-ring pairing identifies $R \otimes_{\mathbb{Z}} A$ with the coinduced module $\mathrm{Hom}_{\mathbb{Z}}(R, A)$ for every abelian group A. Explicitly, $r \otimes a$ corresponds to $s \mapsto \varepsilon_1(sr)a$, where ε_1 takes the coefficient of the identity and the left action on the coinduced module is $(r\varphi)(s) = \varphi(sr)$. The bases $(g)_{g\in\Gamma}$ and $(g^{-1})_{g\in\Gamma}$ are dual for this pairing. Thus every projective R-module is a direct summand of a coinduced module. By the coinduction adjunction, an R-map into such a projective extends along any R-linear injection that splits over $\mathbb{Z}$.

In a finite projective resolution of P of length $d > 0$, the leftmost inclusion $Q_d \to Q_{d-1}$ has $\mathbb{Z}$-free cokernel. The cokernel is P if $d = 1$ and a subgroup of the $\mathbb{Z}$-free module Q_{d-2} otherwise. Extend id_{Q_d} across the inclusion to split it over R. Iteration shortens the resolution to length zero. □

Theorem 3.2. *Let W be a compact Lie group of dimension k with identity component W_0, component group $\Gamma = W/W_0$, and $R = \mathbb{Z}[\Gamma]$. Let $Z \in \mathcal{S}_*^W$ be equivalent to a finite-dimensional based W-CW complex, and let $m \in \mathbb{Z}$. Suppose that:*

(i) *$Z^L \simeq *$ in $\mathcal{S}_*$ for every nontrivial closed subgroup $L \leq W$,*

(ii) *$\widetilde{H}_i(Z; \mathbb{Z}) = 0$ for all $i \notin [m, m + k]$, and*

(iii) *$\widetilde{H}_{m+k}(Z; \mathbb{Z})$ is free abelian.*

Then $P = \widetilde{H}_m(Z; \mathbb{Z})$ is a projective left R-module, and the Pontryagin action of W_0 induces an isomorphism of left R-modules

$$H_t(W_0; \mathbb{Z}) \otimes_{\mathbb{Z}} P \xrightarrow{\cong} \widetilde{H}_{m+t}(Z; \mathbb{Z}) \qquad (t \geq 0), \tag{3.1}$$

where Γ acts diagonally on the left-hand side and via conjugation on $H_t(W_0; \mathbb{Z})$.

Proof. Replace Z by a finite-dimensional based W-CW model.

(a) Let $S \subseteq Z$ consist of $*$ and all cells with nontrivial stabilizer ($K \neq 1$). Since equivariant attaching maps carry points with nontrivial stabilizer to points with nontrivial stabilizer, S is a based W-CW subcomplex of Z. Every cell outside S is free, so $S^L = Z^L \simeq *$ for all $L \neq 1$ by (i). Each cell attachment in S uses objects $(W/K)_+ \wedge E_+$ with $K \neq 1$ and $E = S^{j-1}$ or D^j, for which $\mathrm{Map}_{\mathcal{S}_*^W}((W/K)_+ \wedge E_+, S) \simeq \mathrm{Map}_{\mathcal{S}_*}(E_+, S^K) \simeq *$. The cellular colimit therefore gives $\mathrm{Map}_{\mathcal{S}_*^W}(S, S) \simeq *$ and hence $S \simeq *$ in $\mathcal{S}_*^W$. Thus the quotient map $Z \to T = Z/S$ presents the cofiber and is an equivalence in $\mathcal{S}_*^W$, while the finite disk-skeleton filtration $* = T^{(-1)} \to T^{(0)} \to \cdots \to T^{(M)} = T$ has quotients

$$T^{(s)}/T^{(s-1)} \simeq \bigvee_{\alpha\in I_s} (W_+ \wedge S^s) \qquad (0 \leq s \leq M).$$

(b) Filter the reduced singular chains of T by the disk skeleta, and use integral coefficients throughout. The resulting spectral sequence has differential $d^r : E^r_{s,t} \to E^r_{s-r,t+r-1}$ and converges to $\widetilde{H}_{s+t}(T)$ with a finite filtration. A cycle z of degree q in W acts on the filtered chains by taking the cross product with z on the left and applying the action map. This operation commutes with the differential up to the sign $(-1)^q$, and a boundary acts by a filtered chain homotopy to zero. Thus $H_*(W)$ acts on the spectral sequence and on its abutment. On

$$E^1_{s,t} \cong \bigoplus_{\alpha \in I_s} H_t(W)$$

this is left Pontryagin multiplication on each free-cell summand.

Put $C_s = E^1_{s,0}$ and $\partial_s = d^1$. Since $H_0(W) = R$ acts regularly on each free-cell summand, (C_*, ∂) is a bounded chain complex of free left R-modules in degrees $[0, M]$. Right translation identifies W_0 homeomorphically with each component of W, so left multiplication induces $H_t(W_0) \otimes_{\mathbb{Z}} C_s \xrightarrow{\cong} E^1_{s,t}$ on every column s. Since W_0 is a closed k-manifold, $E^2_{s,t} = 0$ outside $0 \le t \le k$, and for $0 \le t \le k$ we obtain

$$E^2_{s,t} \cong H_s(H_t(W_0) \otimes_{\mathbb{Z}} C_*) \Longrightarrow \widetilde{H}_{s+t}(T), \tag{3.2}$$

where the row-t differential differs from $\mathrm{id} \otimes \partial$ by $(-1)^t$.

(c) Since R is $\mathbb{Z}$-free, each C_s is free abelian. If C_* is acyclic, (3.2) gives $E^2 = 0$, so $\widetilde{H}_*(T) = 0$ and the theorem holds with $P = 0$. Suppose C_* is not acyclic, and let $a \le b$ be the least and greatest degrees in which $H_*(C_*) \ne 0$. The universal coefficient sequence

$$0 \longrightarrow H_t(W_0) \otimes_{\mathbb{Z}} H_s(C_*) \longrightarrow E^2_{s,t} \longrightarrow \mathrm{Tor}^{\mathbb{Z}}_1(H_t(W_0), H_{s-1}(C_*)) \longrightarrow 0 \tag{3.3}$$

shows that $E^2_{s,t} = 0$ outside $a \le s \le b+1$. At $(a, 0)$, outgoing higher differentials land in columns $< a$ and incoming ones originate in rows < 0, so $E^\infty_{a,0} = E^2_{a,0} \cong H_a(C_*) \ne 0$. Hence $\widetilde{H}_a(T) \ne 0$ and $a \ge m$ by (ii). Likewise, orientability of the connected Lie group W_0 gives $H_k(W_0) \cong \mathbb{Z}$ and $E^2_{b,k} \cong H_b(C_*) \ne 0$ by (3.3). Since rows $> k$ and columns $\ge b+2$ vanish on E^2, no higher differentials touch (b, k), so $E^\infty_{b,k} = E^2_{b,k} \ne 0$. Thus $\widetilde{H}_{b+k}(T) \ne 0$ and $b \le m$ by (ii), whence $a = b = m$.

Now E^2 lies in columns m and $m+1$, so $E^2 = E^\infty$. For $k > 0$, Poincaré duality on W_0 gives $H_{k-1}(W_0) \cong H^1(W_0) \cong \mathrm{Hom}_{\mathbb{Z}}(H_1(W_0), \mathbb{Z})$, a torsion-free and hence $\mathbb{Z}$-flat group, so $E^2_{m+1,k-1} = 0$ by (3.3). When $k = 0$, row 0 is the only nonzero row. In either case, the sole associated graded group in total degree $m+k$ is $E^\infty_{m,k} \cong H_m(C_*)$, so $H_m(C_*)$ is free abelian by (iii) and (3.3) kills all of column $m+1$.

In total degree m, the sole surviving filtration piece yields an R-linear isomorphism

$$P \xrightarrow{\cong} \widetilde{H}_m(T) \xrightarrow{\cong} E^\infty_{m,0} = H_m(C_*).$$

In every total degree $m+t$, only $E^\infty_{m,t} \cong H_t(W_0) \otimes_{\mathbb{Z}} H_m(C_*)$ survives, so Pontryagin multiplication induces the isomorphism (3.1), and the identity $[w] \cdot u = (c_w)_*(u) \cdot [w]$ in $H_*(W)$ ($w \in W$, $u \in H_t(W_0)$) makes (3.1) R-linear for the diagonal Γ-action. Finally, put $Z_0 = C_0$ and $Z_s = \ker(\partial_s)$. Exactness of C_* below m splits $0 \to Z_s \to C_s \to Z_{s-1} \to 0$ inductively for $1 \le s \le m$, so Z_m is projective over R and

$$0 \longrightarrow C_M \longrightarrow \cdots \longrightarrow C_{m+1} \longrightarrow Z_m \longrightarrow P \longrightarrow 0$$

is a finite R-projective resolution. By Lemma 3.1, P is projective over R. □

Remark 3.3. The hypotheses of Theorem 3.2 are sharp. Over $W = C_2$ ($k = 0$), the spaces S^m with trivial action, $(EC_2)_+ \wedge S^m$, and $S^1_+ \wedge S^m$ with the antipodal action on S^1 violate the fixed-point, finite-dimensionality, and homology support hypotheses, respectively. Each has $\widetilde{H}_m(Z; \mathbb{Z}) \cong \mathbb{Z}$ with trivial C_2-action, a non-projective $\mathbb{Z}[C_2]$-module since $\mathbb{Z}[C_2]^{C_2}$ has even augmentation. For the second space, the reduced Borel homology is $H_*(BC_2; \mathbb{Z})$ shifted

by m, so it is unbounded. A finite-dimensional model, after the singular collapse in the proof, would have only free cells away from the basepoint and therefore bounded reduced Borel homology. Over $W = S^1$ ($k = 1$, $R = \mathbb{Z}$) with $M = S^m \cup_p D^{m+1}$ ($m \geq 1$, p prime), the space $Z = S^1_+ \wedge M$ has reduced homology $\mathbb{Z}/p$ in degrees m and $m+1$ and 0 elsewhere, yet $\widetilde{H}_m(Z;\mathbb{Z}) = \mathbb{Z}/p$ is not projective over $\mathbb{Z}$.

4. Local Free Cell Attachments

Proposition 4.1. *Let W be a compact Lie group of dimension $k \geq 0$, let A be a finite-dimensional based W-CW complex, let $Y \in \mathcal{S}_*^W$ be equivalent to a finite-dimensional based W-CW complex, and let $f\colon A \to Y$ be a morphism in $\mathcal{S}_*^W$. Assume that:*

(i) *$Y \simeq S^n$ in $\mathcal{S}_*$, where $n \geq k+3$,*
(ii) *$\dim A \leq n-k-1$, and*
(iii) *$f^L\colon A^L \to Y^L$ is an equivalence in $\mathcal{S}_*$ for every nontrivial closed subgroup $L \leq W$.*

Then one can attach free W-cells of disk degree at most $n-k$ to A to obtain a based W-CW complex X and an equivalence $F : X \xrightarrow{\simeq} Y$ in $\mathcal{S}_*^W$ extending f. We have $\dim X = n$ and $X^L = A^L$ for every $L \neq 1$. The cells can be attached in increasing disk degree, with the boundary of each j-cell given by a based map $S^{j-1} \to X^{(j-1)}$ and its W-translates. If Y is a finite-dimensional based W-CW complex and f is a based W-map, F can be taken to extend f as a based W-map into that same model.

Proof. Put $m = n - k \geq 3$. Use the prescribed based W-CW model and map if given. Otherwise choose a finite-dimensional based W-CW model of Y and a based W-map representing f.

Cells below disk degree m. Starting with $B_0 = A$, attach free 1-cells from the basepoint to a chosen vertex in each other path component. For $2 \leq j \leq m-1$, choose based maps $S^{j-1} \to B_{j-1}$ representing generators of $\pi_{j-1}(B_{j-1})$. By based equivariant cellular approximation applied to the adjoint maps from $W_+ \wedge S^{j-1}$, the maps can be taken to land in $B_{j-1}^{(j-1)}$ [9, Theorem (4.4)]. Attach the free j-cells along their W-translates. An ordinary CW structure on W with 1 a vertex shows that $(W \times D^j, W \times S^{j-1})$ contributes ordinary relative cells only in dimensions at least j. The disks $\{1\} \times D^j$ kill the chosen generators, so B_j is $(j-1)$-connected. Each composite $S^{j-1} \to B_{j-1} \to Y$ is nullhomotopic since $j-1 < n$. Taking W-translates of an extension across D^j extends the map to B_j. The same argument for $j = 1$ uses path-connectedness of Y.

Write $g : B \to Y$ for the resulting extension, with $B = B_{m-1}$. Then B is $(m-2)$-connected and has covering dimension at most $n-1$. Let M_g be the reduced mapping cylinder, with B included as the incoming end, and let $C_g = M_g/B$ be the reduced mapping cone. The cylinder is obtained from the ordinary mapping cylinder by collapsing the basepoint interval. A cellular approximation of g gives a finite-dimensional W-CW model of C_g. For $L \neq 1$, the fixed-point homotopy type of C_g is the cofiber of the equivalence $g^L = f^L$. The homology exact sequence gives $\widetilde{H}_q(C_g;\mathbb{Z}) = 0$ for $q < m$. By Illman's restriction theorem, the underlying space of B has an ordinary CW model of dimension at most $n-1$ [6, Theorem A]. Its cellular chains give $H_q(B;\mathbb{Z}) = 0$ for $q \geq n$ and show that $H_{n-1}(B;\mathbb{Z})$ is free abelian. Therefore $\widetilde{H}_q(C_g;\mathbb{Z}) = 0$ for $q > n$, and the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{H}_n(C_g;\mathbb{Z}) \longrightarrow H_{n-1}(B;\mathbb{Z}) \longrightarrow 0$$

shows that $\widetilde{H}_n(C_g;\mathbb{Z})$ is free abelian and nonzero.

Making the bottom homology free. By Theorem 3.2, $P = \widetilde{H}_m(C_g;\mathbb{Z})$ is projective over $R = \mathbb{Z}[\pi_0 W]$. Choose P' with $P \oplus P'$ free and put $Q = (P \oplus P')^{(\mathbb{N})}$, so $P \oplus Q \cong Q$. Add to B one free $(m-1)$-cell for each element of a basis of Q, attaching the entire boundary at the basepoint and extending g constantly on each cell. This wedges copies of $W_+ \wedge S^{m-1}$ onto B and copies of $W_+ \wedge S^m$ onto C_g. The dimension and connectivity bounds for B still hold. The enlarged C_g has free bottom homology over R and still satisfies (3.1), since each

added summand satisfies it by translation on W. Continue to write B, g, M_g, and C_g for the resulting spaces and map.

Cells of disk degree m. The cylinder M_g deformation retracts onto $Y \simeq S^n$. Since B is $(m-2)$-connected, the homotopy exact sequence shows that (M_g, B) is $(m-1)$-connected. The space B is simply connected because $m \geq 3$, and its incoming-end inclusion in M_g is a closed cofibration. Relative Hurewicz and the quotient isomorphism give [8, Theorem 4.37 and Proposition 2.22]

$$\pi_m(M_g, B, *) \xrightarrow{\cong} H_m(M_g, B; \mathbb{Z}) \xrightarrow{\cong} \widetilde{H}_m(C_g; \mathbb{Z}).$$

For each element ξ_μ of an R-basis of the last group, choose a based relative disk $u_\mu : (D^m, S^{m-1}, s_0) \to (M_g, B, *)$ representing its preimage. The boundaries land in $B = B^{(m-1)}$, since all cells of A and B have disk degree at most $m-1$. Attach the free m-cells to B along the W-translates of these boundaries. This gives a based W-CW complex X and a map of pairs $(X, B) \to (M_g, B)$ that is the inclusion on B and is given on the μ-th cell by $(w, z) \mapsto w u_\mu(z)$. Composing with the cylinder retraction gives a based W-map $F : X \to Y$ extending g.

The induced map $X/B \to C_g$ sends each generator in degree m at $1 \in W$ to ξ_μ, by naturality of relative Hurewicz. Hence it is an isomorphism on the free R-modules in degree m. The source is a wedge of copies of $W_+ \wedge S^m$, and both source and target satisfy (3.1). Naturality of Pontryagin multiplication gives an isomorphism in every degree. Comparing the homology sequences of the two pairs and using $M_g \simeq Y$, we conclude that F is an integral homology isomorphism. The new ordinary relative cells have dimension at least m, so X is simply connected like B. The homology isomorphism with the simply connected space Y is therefore a weak equivalence. For $L \neq 1$, the free cells leave $X^L = A^L$ and $F^L = f^L$. This proves that F is an equivalence in $\mathcal{S}_*^W$.

The top homology of the enlarged C_g is nonzero, since the original cone is a wedge summand. By (3.1), its bottom homology is nonzero as well. At least one free m-cell is therefore attached, giving $\dim X = m + k = n$. □

5. Representation Stabilization and the Bauer–FLM Sequence

Let G be a compact Lie group of dimension $d = \dim G$, and let $Y \in (\mathcal{S}_*^G)^\omega$ be a generalized homotopy representation of G. Fix a finite based G-CW complex K_0 of dimension $D = \dim K_0$ of which Y is a retract in $\mathcal{S}_*^G$. Write $\mathrm{Iso}(K_0)$ for the set of point stabilizers of K_0. Since K_0 has finitely many cells, $\mathrm{Iso}(K_0)$ contains only finitely many conjugacy classes of subgroups of G. For every $H \leq G$, we have $\dim K_0^H \leq D$ by (2.3), and restriction to the trivial subgroup of $W_G(H)$ gives an ordinary CW model of K_0^H of dimension at most D [6, Theorem A]. Since $Y^H \simeq S^{n_Y(H)}$ is a retract of K_0^H in $\mathcal{S}_*$, reduced homology gives $n_Y(H) \leq D$.

Lemma 5.1. *There exists a finite-dimensional real orthogonal G-representation U such that:*

(i) $\mathrm{Iso}(K_0) \subseteq \mathrm{Iso}(U)$,

(ii) $\mathrm{Iso}(U)$ *is closed under arbitrary intersections of subgroups, and*

(iii) $\dim U^H - \dim U^L \geq 1$ *whenever* $H \in \mathrm{Iso}(U)$ *and* $L \leq G$ *strictly contains* H.

Proof. Let $H_1, \dots, H_r$ represent the conjugacy classes in $\mathrm{Iso}(K_0)$ ($r \geq 1$ since $* \in K_0^G$), and put $M = G \sqcup \bigsqcup_{i=1}^r G/H_i$, where the regular orbit makes the G-action faithful. Mostow's embedding theorem embeds M G-equivariantly into a real orthogonal G-representation U_0 of dimension $m_0 \geq 1$, so every subgroup in $\mathrm{Iso}(K_0)$ occurs as a vector stabilizer in U_0 [11, Theorem 6.1]. Put $U = U_0^{\oplus m_0}$. The stabilizer of $(v_1, \dots, v_{m_0}) \in U$ is the pointwise stabilizer G_E of $E = \mathrm{span}_{\mathbb{R}}(v_1, \dots, v_{m_0}) \subseteq U_0$, and since $\dim U_0 = m_0$, every linear subspace $E \subseteq U_0$ arises in this way. Thus $\mathrm{Iso}(U) = \{G_E : E \subseteq U_0 \text{ linear}\}$ contains $\mathrm{Iso}(U_0) \supseteq \mathrm{Iso}(K_0)$, and $\bigcap_\alpha G_{E_\alpha} = G_{\sum_\alpha E_\alpha} \in \mathrm{Iso}(U)$. Whenever $H \in \mathrm{Iso}(U)$ and $H < L$, pick $u \in U$ with $G_u = H$. Then $u \in U^H \setminus U^L$, so $\dim U^H - \dim U^L \geq 1$. □

Fix an integer $q \geq D + 2d + 1$, put $V = U^{\oplus q} \oplus \mathbb{R}^{d+3}$ with trivial G-action on $\mathbb{R}^{d+3}$, and set $\widetilde{Y} = Y \wedge S^V \in (\mathcal{S}_*^G)^\omega$, so that $\widetilde{Y}^H \simeq S^{n(H)}$ with $n(H) = n_Y(H) + \dim V^H$. Set $\mathcal{I} = \mathrm{Iso}(U) = \mathrm{Iso}(V) = \mathrm{Iso}(S^V)$. Since S^V admits a finite G-CW structure (Lemma 2.2), $\mathcal{I}$ consists of finitely many conjugacy classes.

Lemma 5.2. *With the notation above:*

(i) *$n(H) \geq d + 3$ for every $H \leq G$, and $n(H) - n(L) \geq 2d + 1$ whenever $H \in \mathcal{I}$ and $H < L$.*

(ii) *Every closed subgroup $L \leq G$ is contained in a unique least subgroup $M_L \in \mathcal{I}$, and restriction along $L \leq M_L$ induces an equivalence $\widetilde{Y}^{M_L} \xrightarrow{\simeq} \widetilde{Y}^L$ in $\mathcal{S}_*$, while $X^{M_L} = X^L$ for any based G-CW complex X whose cell stabilizers lie in $\mathcal{I}$.*

Proof. The trivial summand $\mathbb{R}^{d+3}$ gives $n(H) \geq d + 3$, and for $H \in \mathcal{I}$ with $H < L$, Lemma 5.1(iii) gives

$$n(H) - n(L) = n_Y(H) - n_Y(L) + q(\dim U^H - \dim U^L) \geq -D + q \geq 2d + 1,$$

and (i) follows. For (ii), $M_L = \bigcap\{K \in \mathcal{I} : L \leq K\}$ lies in $\mathcal{I}$ by Lemma 5.1(ii). Any point of X fixed by L has stabilizer in $\mathcal{I}$ containing L and hence containing M_L, so $X^{M_L} = X^L$. Away from the basepoint (whose stabilizer is $G \in \mathcal{I}$), every point stabilizer of $K_0 \wedge S^V$ has the form $G_z \cap G_v \in \mathcal{I}$ by Lemma 5.1(i)–(ii), so $(K_0 \wedge S^V)^{M_L} = (K_0 \wedge S^V)^L$. Since $\widetilde{Y}$ is a retract of $K_0 \wedge S^V$ in $\mathcal{S}_*^G$, the map $\widetilde{Y}^{M_L} \to \widetilde{Y}^L$ is a retract of an identity map in $\mathcal{S}_*$ and is therefore an equivalence. □

Proof of Theorem 1.2. Choose a finite-dimensional based G-CW complex T representing $\widetilde{Y}$ by Lemma 2.1. Then T^H is a finite-dimensional based $W_G(H)$-CW complex for each H. By Lemma 5.2(ii), the inclusion $T^{M_L} \to T^L$ is a homotopy equivalence for every L.

Order the conjugacy classes in $\mathcal{I}$ as $(H_1), \dots, (H_N)$ so that a class strictly above another under subconjugacy occurs first. This is possible since inclusion between conjugate closed subgroups is equality, by dimension and component count. We construct a based G-CW complex X_a and a based G-map $F_a : X_a \to T$ at each stage. Its cells have isotropy types among the first a classes, and cells of type (K) have disk degree at most $n(K) - \dim W_G(K)$. On the first a types, F_a^K is a homotopy equivalence and $\dim X_a^K = n(K)$. For $(H_1) = (G)$, take $X_1 = S^{n(G)}$ with trivial action and choose a based equivalence to T^G for F_1.

Suppose $a > 1$ and put $H = H_a$, $W = W_G(H)$, $k = \dim W$, and $m = n(H) - k \geq 3$. Let $X_{\mathrm{old}} = X_{a-1}$. Each old cell $G/K \times D^j$ with nonempty H-fixed set has, after conjugation, $H < K$. Its H-fixed part has geometric dimension at most

$$j + \dim(G/K)^H \leq n(K) + d \leq n(H) - k - 1,$$

using Lemma 5.2(i) and $k \leq d$. Thus $\dim X_{\mathrm{old}}^H \leq m - 1$. For a nontrivial $L \leq W$, let $\widetilde{L}$ be its preimage in $N_G(H)$. The class of $M_{\widetilde{L}}$ has already been treated because $H < \widetilde{L} \leq M_{\widetilde{L}}$. We have $X_{\mathrm{old}}^{\widetilde{L}} = X_{\mathrm{old}}^{M_{\widetilde{L}}}$ and $T^{M_{\widetilde{L}}} \simeq T^{\widetilde{L}}$ by Lemma 5.2(ii). Consequently $(F_{a-1}^H)^L = F_{a-1}^{\widetilde{L}}$ is an equivalence.

Apply Proposition 4.1 to the based W-map $X_{\mathrm{old}}^H \to T^H$ and use its cellular extensions in order of disk degree. At degree j, let Z denote the entire current G-complex and let $f_Z : Z \to T$ be its current map. For each local free j-cell, write $a : S^{j-1} \to (Z^H)^{(j-1)}$ for its boundary at $1 \in W$ and $h : D^j \to T^H$ for the disk map, so $h|_{S^{j-1}} = f_Z^H a$. Attach a $G/H \times D^j$ to Z by $(gH, s) \mapsto ga(s)$ and extend f_Z by $(gH, z) \mapsto gh(z)$. The maps are well-defined because H fixes both images. By (2.2) the boundary lies in $Z^{(j-1)}$, so the attachment preserves the absolute CW structure, including the higher cells of Z. Since $(G/H)^H = W$, taking H-fixed points recovers exactly the local attachments and their maps. For any previously treated K, $(G/H)^K = \varnothing$, so its fixed space and map are unchanged.

After all these attachments, the resulting $F_a : X_a \to T$ has the two required properties, including the dimension equality at H by Proposition 4.1. At $a = N$, set $X = X_N$. For any $L \leq G$, we have $X^L = X^{M_L}$ and $T^{M_L} \simeq T^L$. Hence F_N^L is a homotopy equivalence and

$\dim X^L = n(M_L) = n(L)$. The fixed-point equivalences give $X \simeq T \simeq Y \wedge S^V$ in $\mathcal{S}^G_*$. In particular X is compact and its dimension equalities make it a homotopy representation. □

Proof of Theorem 1.1. For every real G-representation V, the class $[S^V]$ lies in $V(G)$ (Lemma 2.2). Given a generalized homotopy representation $Y \in (\mathcal{S}^G_*)^\omega$, Theorem 1.2 supplies a homotopy representation X with $X \simeq Y \wedge S^V$ in $\mathcal{S}^G_*$, so $[Y] = \rho_G([X] - [S^V])$ in $V'(G)$ and ρ_G is surjective. Suppose $\rho_G([A] - [B]) = 0$ for homotopy representations A and B. Group completion provides a generalized homotopy representation Z with $A \wedge Z \simeq B \wedge Z$ in $\mathcal{S}^G_*$. Choose a homotopy representation $Z_0 \simeq Z \wedge S^U$ by Theorem 1.2. Smash product with S^U gives $A \wedge Z_0 \simeq B \wedge Z_0$ in $\mathcal{S}^G_*$, so $[A] = [B]$ in $V(G)$ (compare [7, Lemma 5.5]). Hence ρ_G is an isomorphism, and σ_G is an isomorphism by [3, Proposition 4.6]. □

Proof of Corollary 1.3. The map $c\colon \mathrm{Pic}(A(G)) \to \mathrm{Pic}(\mathrm{Sp}^G)$ is injective with image equal to the kernel of the stable dimension homomorphism $\widetilde{d}\colon \mathrm{Pic}(\mathrm{Sp}^G) \to C(G)$ defined via geometric fixed points $\Phi^H\colon \mathrm{Sp}^G \to \mathrm{Sp}$ [3, Theorems 0.1, 0.2, and 3.2]. Since $\Phi^H \Sigma^\infty_G X \simeq \Sigma^\infty X^H$ in Sp [3, §2 and Definition 3.1], the composite $\widetilde{d} \circ \sigma_G \circ \rho_G$ sends $[X] - [Y] \in V(G)$ to $n_X - n_Y$. Bauer works with unbased homotopy representations under join, where the H-fixed homotopy sphere has dimension $d_X(H) - 1$ ($S^{-1} = \varnothing$) [1, Definition 2.1 and Theorem 2.4]. Unreduced suspension sends $S^{d_X(H)-1}$ to $S^{d_X(H)}$, matches join with based smash product, and identifies $D(G) \subseteq C(G)$ with the image of $[X] - [Y] \mapsto n_X - n_Y$ on $V(G)$ [3, Definition 4.3]. Theorem 1.1 and [1, Theorem 2.4] therefore give $\mathrm{im}(\widetilde{d}) = D(G)$, so the corestriction d is surjective with kernel $\mathrm{im}(c)$. □

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